Improving Space Localization Properties of the Discrete Wavelet Transform

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Abstract. In this paper, a modified version of the discrete wavelet transform (DWT), distinguishing itself with visibly improved space localization properties and noticeably extended potential capabilities, is proposed. The key point of this proposal is the full decorrelation of wavelet coefficients across the lower scales. This proposal can be applied to any DWT of higher orders (Le Gall, Daubechies D4, CDF 9/7, etc.). To open up new areas of practical applicability of the modified DWT, a novel exceptionally fast algorithm for computing the DWT spectra of the selected signal (image) blocks is presented. In parallel, some considerations and experimental results concerning the energy compaction property of the modified DWT are discussed.

Key words: wavelets, discrete wavelet transforms, lifting schemes, decorrelation of wavelet coefficients, image processing

1. Introduction

The discrete wavelet transform (DWT) has a huge number of applications in science, engineering, mathematics and computer science. Surely, this is conditioned not only by attractiveness of wavelet properties,such as compact support, dilating relation, vanishing moments, smoothness, etc., but also by fairly easy access to computer software packages that include fast and efficient algorithms to perform wavelet transforms.

First and foremost, wavelets have been applied in computer science research areas such as signal processing (time series analysis, noise reduction, feature extraction, texture analysis and classification, etc. (Donoho and Johnstone, 1995; Perlibakas, 2004; Zhang *et al.*, 2005; Molavi and Sadr, 2007; Liu *et al.*, 2009; Chaovalit *et al.*, 2011; Kaganami *et al.*, 2011), data (digital signal, image, video) compression (Shapiro, 1993; Said and Pearlman, 1996; Christopoulos *et al.*, 2000; Jensen and Cour-Harbo, 2001; Vetterli, 2001; Sudhakaret *et al.*, 2005; Song, 2006; Valantinas and Kančelkis, 2010), data hiding techniques for

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images (Lai and Chang, 2006; Yang *et al.*, 2012), data mining (data management, data mining tasks and algorithms, etc. (Shahabi *et al.*, 2000; Hand *et al.*, 2001; Prabakaran *et al.*, 2006), and computer graphics (multi-resolution data representation, multi-resolution rendering, etc. (Stollnitz *et al.*, 1996; Lin and He, 2011).

Furthermore, for control applications, the wavelet transform is used in motion tracking, robot positioning, identification and both linear and nonlinear control purposes (Purwaret et al., 2007; Nandhakumar and Selladurai, 2011). Finally, wavelets are a powerful tool for the analysis and adjustment of audio signals (Darlington *et al.*, 2002; Kumar *et al.*, 2010).

Wavelet analysis is an indispensable technique for the study of algorithms that use numerical approximation and can be highly computationally efficient, in the first instance for solving ordinary and partial differential equations (Mehra and Kevlahan, 2008).

The distinguishing feature of the wavelet transform is that individual basis functions are localized in space. Full localization in space means that numerical values of wavelet coefficients at the same scale are specified exclusively by signal (image) blocks that cover the whole signal (image) and do not overlap. Unfortunately, only the simplest Haar wavelet transform (Rao and Bopadikar, 2002; Phang and Phang, 2008), most frequently used as an educational study, is fully localized in space. Higher order wavelet (Le Gall, Daubechies D4, CDF 9/7, etc.) transforms (Jensen and Cour-Harbo, 2001; Rao and Bopadikar, 2002) are partially localized in space since the signal (image) blocks which specify numerical values of adjacent wavelet coefficients at the same scale overlap. This results in some limitations to the practical application of higher order wavelet transforms, especially in such areas where block-processing of digital data is preferable, e.g. the image pattern recognition, the signal (image) feature extraction, localization of defects in textural images, locally progressive image encoding, etc.

In this paper, a new modified version of the discrete wavelet transform (DWT) that distinguishes itself with apparently improved space localization properties is proposed. The essence of this proposal is a full decorrelation of wavelet coefficients across the lower scales. To show the extended potential capabilities of the proposed version of DWT, an alternative faster algorithm for computing the DWT spectra of the selected signal (image) blocks is presented. The latter algorithm explores the space localization properties intrinsic to the modified version of DWT.

The rest of the paper is organized as follows. Section 2 introduces the discrete wavelet transform and describes the procedures for finding the discrete wavelet coefficients of a digital signal (image). Section 3 deals with factors ensuring the improved space localization properties of the DWT and proposes a novel fast algorithm for computing the DWT spectra of the selected signal (image) blocks, thereby giving rise to new real-time applications of the discrete wavelet transform. Some commentary on the energy compaction property of the developed version of DWT and the experimental analysis results are presented in Section 4.

2. Implementing the Discrete Wavelet Transform

The one-dimensional discrete wavelet transform (DWT) itself represents an iterative procedure, each iteration of the DWT applies the scaling function to the data input. If the

original data input (digital signal) X has N ($N = 2^n$, $n \in \mathbb{N}$) values, the scaling function will be applied in the wavelet transform iteration to calculate $N/2$ averaged (smoothed) values. In the ordered wavelet transform the smoothed values are stored in the upper half of the N element input vector. The wavelet function, for each iteration of the wavelet transform, is also applied to the data input. If the original signal has N values, the wavelet function will be applied to calculate $N/2$ differences that reflect changes in the data. In the ordered wavelet transform, the differenced values are stored in the lower half of the N element input vector. On the next iteration, both the scaling and the wavelet functions are applied repeatedly to the ordered set of smoothed values calculated during the preceding iteration. After a finite number of iterations (n steps) the DWT spectrum Y of the digital signal X is found. The vector Y comprises the only smoothed value, obtained in the n-th iteration, and the ordered set of differenced values, obtained in the $n-1$ preceding iterations.

It should be noted that the inverse DWT is obtained by applying the inverse of each of the above mentioned steps in the opposite order.

Now, let $S^{(i)} = (s_0^{(i)} s_1^{(i)} s_2^{(i)} \ldots s_{2^{n-i}-1}^{(i)})^T$ and $D^{(i)} = (d_0^{(i)} d_1^{(i)} d_2^{(i)} \ldots d_{2^{n-i}-1}^{(i)})^T$ be the result of the application of the DWT scaling and wavelet functions to the input or intermediate data vector $S^{(i-1)}$ ($i \in \{1, 2, \ldots, n\}$), respectively (here, $S^{(0)} = X = (x_0 x_1 x_2 \ldots x_{N-1})^T$, i.e. $s_k^{(0)} = x_k$, for all $k = 0, 1, \ldots, N - 1$). The discrete DWT spectrum Y of the data vector X is obtained in n iterations and takes the form:

$$
Y = \big(s_0^{(n)} d_0^{(n)} d_0^{(n-1)} d_1^{(n-1)} d_0^{(n-2)} d_1^{(n-2)} d_2^{(n-2)} d_3^{(n-2)} d_0^{(n-3)} d_1^{(n-3)} \ldots
$$
$$
d_0^{(1)} d_1^{(1)} \ldots d_{N/2-1}^{(1)} \big)^T. \tag{1}
$$

Theoretically, the i-th ($i \in \{1, 2, \ldots, n\}$) iteration of the above computational procedure can be realized using the DWT matrix T_{DWT} of order $(n - i + 1)$, i.e.

$$
\begin{pmatrix} S^{(i)} \\ D^{(i)} \end{pmatrix} = T_{\text{DWT}}(n - i + 1) \cdot S^{(i-1)}. \tag{2}
$$

Here we observe that $S^{(n)} = (s_0^{(n)})$ and $D^{(n)} = (d_0^{(n)})$.

The internal structure of the matrix $T_{\text{DWT}} = T_{\text{DWT}}(n - i + 1)$, $i \in \{1, 2, \ldots, n\}$, depends, mainly, on the coefficient values of scaling and wavelet functions of a particular DWT (expression (2)). For instance, in the case of the discrete Haar wavelet transform (HT), the scaling function coefficients are $- h_0 = 1/\sqrt{2}$ and $h_1 = 1/\sqrt{2}$, while the wavelet function coefficient values are $- g_0 = h_1 = 1/\sqrt{2}$ and $g_1 = -h_0 = -1/\sqrt{2}$. So, the HT matrices take the form:

$$
T_{\text{HT}}(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
$$

$$
T_{\text{HT}}(2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},
$$

$$T_{\text{HT}}(3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

and so on.

On the other hand, the discrete Le Gall wavelet transform (DLGT) has five scaling and three wavelet function coefficients. The scaling function coefficients are – $h_0 = -0.125$, $h_1 = 0.25$, $h_2 = 0.75$, $h_3 = 0.25$ and $h_4 = -0.125$, whereas the wavelet function coefficient values are – $g_0 = -0.5$, $g_1 = 1$ and $g_2 = -0.5$. Correspondingly, the DLGT matrices have the form:

$$T_{\text{DLGT}}(1) = \begin{array}{cc} -0.125 & 0.25 \\ \end{array} \begin{pmatrix} 0.75 & 0.25 \\ -0.5 & 1 \end{pmatrix} \begin{array}{c} -0.125 \\ -0.5 \end{array},$$

$$T_{\text{DLGT}}(2) = \begin{array}{cc} -0.125 & 0.25 \\ \end{array} \begin{pmatrix} 0.75 & 0.25 & -0.125 & 0 \\ -0.125 & 0.25 & 0.75 & 0.25 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & 0 & -0.5 & 1 \end{pmatrix} \begin{array}{c} -0.125 \\ \\ -0.5 \end{array},$$

$T_{\text{DLGT}}(3)$

$$= \begin{array}{cc} -0.125 & 0.25 \\ \end{array} \begin{pmatrix} 0.75 & 0.25 & -0.125 & 0 & 0 & 0 & 0 & 0 \\ -0.125 & 0.25 & 0.75 & 0.25 & -0.125 & 0 & 0 & 0 \\ 0 & 0 & -0.125 & 0.25 & 0.75 & 0.25 & -0.125 & 0 \\ 0 & 0 & 0 & 0 & -0.125 & 0.25 & 0.75 & 0.25 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 1 \end{pmatrix} \begin{array}{c} -0.125 \\ \\ \\ -0.5 \end{array}$$

and so on.

It is easy to see that the matrices $T_{\text{DLGT}}(n)$ ($n = 1, 2, 3, \ldots$) contain some elements which lie outside the matrix. Keeping in mind that the matrices $T_{\text{DLGT}}(n)$ are applied to the data vectors of size $N = 2^n$, at both ends of the data vector the so-called "edge" problem occurs. Several approaches to handling the "edge" problem are practised, namely: (1) treating the data vector as if it were periodic; (2) treating the data vector as if it were mirrored at the ends; (3) calculating special scaling and wavelet functions that are applied at the start and end of the data vector (Gram-Schmidt orthogonalization).

In the case of the discrete Daubechies D4 wavelet transform (D4), we have: $h_0 = (1 + \sqrt{3})/4\sqrt{2}$, $h_1 = (3 + \sqrt{3})/4\sqrt{2}$, $h_2 = (3 - \sqrt{3},)/4\sqrt{2}$, $h_3 = (1 - \sqrt{3})/4\sqrt{2}$, for the scaling function, and $g_0 = h_3$, $g_1 = -h_2$, $g_2 = h_1$, $g_3 = -h_0$, for the wavelet function.

TheD4 matrices are of the form:

$$T_{D4}(1) = \begin{pmatrix} h_0 & h_1 \\ g_0 & g_1 \end{pmatrix} \begin{matrix} h_2 & h_3 \\ g_2 & g_3 \end{matrix},$$

$$T_{D4}(2) = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 \\ 0 & 0 & h_0 & h_1 \\ g_0 & g_1 & g_2 & g_3 \\ 0 & 0 & g_0 & g_1 \end{pmatrix} \begin{matrix} h_2 h_3 \\ \\ g_2 g_3 \end{matrix},$$

$$T_{D4}(3) = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_0 & h_1 \\ g_0 & g_1 & g_2 & g_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_0 & g_1 & g_2 & g_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_0 & g_1 \end{pmatrix} \begin{matrix} h_2 h_3 \\ \\ \\ \\ \\ \\ \\ g_2 g_3. \end{matrix}$$

and so on. As it can be seen, the "edge" problem manifests itself again.

To become familiarized with some other discrete wavelet transforms (integer-to-integer Le Gall wavelet transform, Cohen-Daubechies-Feauveau (CDF) 9/7, etc.) and their matrices, you can refer to (Cohen *et al.*, 1992; Adams and Kossentine, 2000; open internet resources).

In practice, to compute the DWT spectrum Y (expression (1)) of the data vector X of size $N = 2^n$ ($n \in \mathbb{N}$), efficient procedures, called Lifting Schemes, are applied (Sweldens, 1997). For instance, in the case of HT, the discrete spectrum Y is found as follows:

$$s_k^{(i)} = h_0 \cdot s_{2k}^{(i-1)} + h_1 \cdot s_{2k+1}^{(i-1)} = \frac{1}{\sqrt{2}} \big(s_{2k}^{(i-1)} + s_{2k+1}^{(i-1)} \big), \tag{3}$$

$$d_k^{(i)} = g_0 \cdot s_{2k}^{(i-1)} + g_1 \cdot s_{2k+1}^{(i-1)} = \frac{1}{\sqrt{2}} \big(s_{2k}^{(i-1)} - s_{2k+1}^{(i-1)} \big), \tag{4}$$

for all $k = 0, 1, \ldots, 2^{n-i} - 1$ and $i \in \{1, 2, \ldots, n\}$.

The inverse HT is defined by:

$$s_{2k}^{(i-1)} = \frac{1}{\sqrt{2}} \big(s_k^{(i)} + d_k^{(i)} \big), \qquad s_{2k+1}^{(i-1)} = \frac{1}{\sqrt{2}} \big(s_k^{(i)} - d_k^{(i)} \big), \tag{5}$$

where $k = 0, 1, \ldots, 2^{n-i} - 1$ and $i \in \{1, 2, \ldots, n\}$.

To compute the discrete DLGT spectrum Y of the data vector X of size $N = 2^n$ ($n \in \mathbb{N}$), the following fast procedure (Lifting Scheme) is applied:

$$d_k^{(i)} = s_{2k+1}^{(i-1)} - \frac{1}{2} \big(s_{2k}^{(i-1)} + s_{2k+2}^{(i-1)} \big), \qquad s_k^{(i)} = s_{2k}^{(i-1)} + \frac{1}{4} \big(d_{k-1}^{(i)} + d_k^{(i)} \big), \tag{6}$$

for all $k = 0, 1, \ldots, 2^{n-i} - 1$; here $s_{2^{n-i+1}}^{(i-1)} := s_{2^{n-i+1}-2}^{(i-1)}, d_{-1}^{(i)} := d_0^{(i)}$ and $i \in \{1, 2, \ldots, n\}$.

The "edge" problem, which takes place at both ends of the data vector and which determines the partially localized nature of the discrete Le Gall wavelet transform, here(expression (6)) is solved by treating the data vectors $S^{(i)}$ ($i \in \{0, 1, \ldots, n-1\}$) as if they were mirrored at the ends.

The inverse DLGT is specified by:

$$s_{2k}^{(i-1)} = s_k^{(i)} - \frac{1}{4}\left(d_{k-1}^{(i)} + d_k^{(i)}\right), \qquad s_{2k+1}^{(i-1)} = d_k^{(i)} + \frac{1}{2}\left(s_{2k}^{(i-1)} + s_{2k+2}^{(i-1)}\right), \tag{7}$$

for all $k = 0, 1, \ldots, 2^{n-i} - 1$ and $i \in \{1, 2, \ldots, n\}$.

The Lifting Scheme used to find the discrete D4 spectrum of the data vector X of size $N = 2^n$ ($n \in \mathrm{N}$) is presented below. For all $i \in \{1, 2, \ldots, n\}$ and $k = 0, 1, \ldots, 2^{n-i} - 1$, the following steps are realized:

1. $\tilde{s}_k^{(i)} = s_{2k}^{(i-1)} + \sqrt{3}s_{2k+1}^{(i-1)}, \qquad \tilde{d}_k^{(i)} = \frac{\sqrt{3}}{4}\tilde{s}_{k+1}^{(i)} - \frac{2-\sqrt{3}}{4}\tilde{s}_k^{(i)} - s_{2k+3}^{(i-1)}, \quad$ where

   $$s_{2(2^{n-i}-1)+3}^{(i-1)} = s_{2^{n-i+1}+1}^{(i-1)} := s_1^{(i-1)}, \quad \tilde{s}_{2^{n-i}}^{(i)} := \tilde{s}_0^{(i)}.$$

2. $s_k^{(i)} = \frac{\sqrt{3}-1}{\sqrt{2}}\left(\tilde{s}_k^{(i)} + \tilde{d}_k^{(i)}\right), \qquad d_k^{(i)} = \frac{\sqrt{3}+1}{\sqrt{2}}\tilde{d}_k^{(i)}. \tag{8}$

Here, the "edge" problem is solved by treating the data vectors $S^{(i)}$ ($i \in \{0, 1, \ldots, n-1\}$) as if they were periodic.

The inverse D4 transform is specified, for all $i \in \{1, 2, \ldots, n\}$ and $k = 0, 1, \ldots, 2^{n-i} - 1$, as follows:

1. $\tilde{d}_k^{(i)} = \frac{\sqrt{3}-1}{\sqrt{2}}d_k^{(i)}, \qquad \tilde{s}_k^{(i)} = \frac{\sqrt{3}+1}{\sqrt{2}}s_k^{(i)} - \tilde{d}_k^{(i)}.$

2. $s_{2k}^{(i-1)} = \tilde{s}_k^{(i)} - \sqrt{3}s_{2k+1}^{(i-1)}, \qquad s_{2k+1}^{(i-1)} = \frac{\sqrt{3}}{4}\tilde{s}_k^{(i)} - \frac{2-\sqrt{3}}{4}\tilde{s}_{k-1}^{(i)} - \tilde{d}_{k-1}^{(i)}, \tag{9}$

where $\tilde{s}_{-1}^{(i)} := \tilde{s}_{2^{n-i}-1}^{(i)}$ and $\tilde{d}_{-1}^{(i)} := \tilde{d}_{2^{n-i}-1}^{(i)}$.

The more detailed material on the development of Lifting Schemes for various DWT (expressions (3)–(9)), can be found in a tutorial (Sweldens, 1997), and some interesting ideas behind lifting – in the books (Jensen and Cour-Harbo, 2001; Van Fleet, 2006).

Also, in order to compute the DWT spectrum of a two-dimensional digital image X of size $N_1 \times N_2$ ($N_i = 2^{n_i}$, $n_i \in \mathrm{N}$, $i = 1, 2$), the one-dimensional DWT should be applied $N_1 + N_2$ times, i.e. N_2 times along the first spatial axis and N_1 times along the second spatial axis.

3. Extending Potential Capabilities of the Discrete Wavelet Transform

The characteristic feature of the discrete wavelet transform (DWT) is that individual basis functions are localized in space. As it was mentioned above, full localization in space (or the equivalent full decorrelation of wavelet coefficients) means that numerical values of wavelet coefficients that are computed at the same scale are specified exclusively by signal (image) blocks that cover the whole signal (image) and do not overlap. Unfortunately, only the simplest Haar wavelet transformis fully localized in space, higher order wavelet transforms (Le Gall, Daubechies D4, CDF 9/7, etc. Jensen and Cour-Harbo, 2001; Rao and Bopadikar, 2002) are partially localized in space since signal (image) blocks which specify numerical values of adjacent wavelet coefficients overlap. This gives some limitations to the practical application of higher order DWT, especially in such areas where block-processing of digital data is foreseen, e.g. the signal (image) feature extraction, locally progressive signal (image) encoding, localization of defects in textural images, etc.

3.1. *Full Decorrelation of Wavelet Coefficients at Lower Scales*

In the paper, full decorrelation of DWT spectral coefficients across the lower scales, leading to notably improved space localization properties of the transform, is achieved by partitioning the original data vector (digital signal) $X = S^{(0)}$ of size N ($N = 2^n$, $n \in \mathrm{N}$) into a set of 2^{n-p} non-overlapping blocks $S_j^{(0)}$ ($j = 0, 1, \ldots, 2^{n-p} - 1$) of a previously prescribed size 2^p ($1 \leqslant p < n$) and by transferring the earlier mentioned "edge" problem (Section 2) to those blocks.

Computing the modified DWT spectrum Y of $X = S^{(0)}$ is carried out in accordance with the generalized iterative procedure presented below:

(a) If $i \in \{1, 2, \ldots, p\}$, then

$$\left(S_j^{(i-1)} \right) \xrightarrow{T_{\mathrm{DWT}}(p-i+1)} \begin{pmatrix} S_j^{(i)} \\ D_j^{(i)} \end{pmatrix}, \quad \text{for all } j = 0, 1, \ldots, 2^{n-p} - 1;$$

(b) If $i \in \{p + 1, p + 2, \ldots, n\}$, then

$$\begin{pmatrix} S_{2j}^{(i-1)} \\ S_{2j+1}^{(i-1)} \end{pmatrix} \xrightarrow{T_{\mathrm{DWT}}(1)} \begin{pmatrix} S_j^{(i)} \\ D_j^{(i)} \end{pmatrix}, \quad \text{for all } j = 0, 1, \ldots, 2^{n-i} - 1.$$

Evidently, $S_j^{(i)} = (s_j^{(i)})$, $D_j^i = (d_j^{(i)})$, for all $i \in \{p, p+1, \ldots, n\}$ and $j = 0, 1, \ldots, 2^{n-i} - 1$.

As it can be seen, in the above scheme, not only the non-overlapping component blocks $S_j^{(0)}$ ($j = 0, 1, \ldots, 2^{n-p} - 1$) of the original data vector X but also non-overlapping component blocks $S_j^{(i)}$ ($j = 0, 1, \ldots, 2^{n-p-i} - 1$) of the intermediate data vectors $S^{(i)}$ ($i \in \{0, 1, \ldots, p - 1\}$), as well as data vectors $(s_{2j}^{(i)} s_{2j+1}^{(i)})^T$ ($i \in \{p, p + 1, \ldots, n - 1\}$; $j = 0, 1, \ldots, 2^{n-i} - 1$), are processed by treating them as if they were periodic or mirrored at the ends.

This way, the improved full localization in space is ensured for all wavelet coefficients across the i-th ($i \in \{p, p+1, \ldots, n\}$) scale.

The modified DWT spectrum Y, as before (Section 2), is obtained in n iterations and can be represented in the form

$$Y = (s_0^{(n)} d_0^{(n)} d_0^{(n-1)} d_1^{(n-1)} d_0^{(n-2)} d_1^{(n-2)} d_2^{(n-2)} d_3^{(n-2)} \ldots d_0^{(1)} d_1^{(1)} \ldots d_{N/2-1}^{(1)})^T.$$

Following this insight, we have developed new iterative procedures (Lifting Schemes) for computing the modified DWT spectrafor one-dimensional digital signals.

In the case of DLGT, the developed iterative procedure (Lifting Scheme) is of the form (Valantinas, 2009):

$$d_k^{(i)} = \begin{bmatrix} s_{2k+1}^{(i-1)} - s_{2k}^{(i-1)}, & k \in \{l_i - 1, 2l_i - 1, \ldots, q_i l_i - 1\}, \\ s_{2k+1}^{(i-1)} - \frac{1}{2}(s_{2k}^{(i-1)} + s_{2k+2}^{(i-1)}), & \text{otherwise,} \end{bmatrix} \tag{10}$$

$$s_k^{(i)} = \begin{bmatrix} s_{2k}^{(i-1)} + \frac{1}{2}d_k^{(i)}, & k \in \{0, l_i, 2l_i, \ldots, (q_i - 1)l_i\}, \\ s_{2k}^{(i-1)} + \frac{1}{4}(d_{k-1}^{(i)} + d_k^{(i)}), & \text{otherwise,} \end{bmatrix} \tag{11}$$

for all $k = 0, 1, \ldots, 2^{n-i} - 1$ and $i \in \{1, 2, \ldots, n\}$; here: $l_i = 2^{p-i}$, $q_i = 2^{n-p}$, for $i = 1, 2, \ldots, p$, and $l_i = 1$, $q_i = 2^{n-i}$, for $i = p+1, p+2, \ldots, n$. Evidently, $l_i q_i$ equals the size of $S^{(i)}$, $i = 1, 2, \ldots, n$.

A new modified lifting scheme for the inverse DLGT has also been developed (Valantinas, 2009), namely:

$$s_{2k}^{(i-1)} = \begin{bmatrix} s_k^{(i)} - \frac{1}{2}d_k^{(i)}, & k \in \{0, l_i, 2l_i, \ldots, (q_i - 1)l_i\}, \\ s_k^{(i)} - \frac{1}{4}(d_{k-1}^{(i)} + d_k^{(i)}), & \text{otherwise,} \end{bmatrix} \tag{12}$$

$$s_{2k+1}^{(i-1)} = \begin{bmatrix} d_k^{(i)} + s_{2k}^{(i-1)}, & k \in \{l_i - 1, 2l_i - 1, \ldots, q_i l_i - 1\}, \\ d_k^{(i)} + \frac{1}{2}(s_{2k}^{(i-1)} + s_{2k+2}^{(i-1)}), & \text{otherwise,} \end{bmatrix} \tag{13}$$

for all $k = 0, 1, \ldots, 2^{n-i} - 1$ and $i \in \{1, 2, \ldots, n\}$.

The modified Lifting Scheme used to find the discrete modified Daubechies D4 spectrum of the digital signal X of size $N = 2^n$ ($n \in \mathbb{N}$) is presented below.

For all $k = 0, 1, \ldots, 2^{n-i} - 1\}$ ($i \in \{1, 2, \ldots, n\}$), the following steps are realized:

1. $\tilde{s}_k^{(i)} = s_{2k}^{(i-1)} + \sqrt{3}s_{2k+1}^{(i-1)}$, $\tilde{d}_k^{(i)} = \frac{\sqrt{3}}{4}\tilde{s}_{k+1}^{(i)} - \frac{2-\sqrt{3}}{4}\tilde{s}_k^{(i)} - s_{2k+3}^{(i-1)}$,

where: $s_{2k+3}^{(i-1)} := s_{2(k-l_i)+3}^{(i-1)}$, $\tilde{s}_{k+1}^{(i)} := \tilde{s}_{k-l_i+1}^{(i)}$, for $k \in \{l_i - 1, 2l_i - 1, \ldots, q_i l_i - 1\}$; $l_i = 2^{p-i}$, $q_i = 2^{n-p}$, for $i = 1, 2, \ldots, p$, and $l_i = 1$, $q_i = 2^{n-i}$, for $i = p+1, p+2, \ldots, n$.

2. $s_k^{(i)} = \frac{\sqrt{3}-1}{\sqrt{2}}(\tilde{s}_k^{(i)} + \tilde{d}_k^{(i)})$, $d_k^{(i)} = \frac{\sqrt{3}+1}{\sqrt{2}}\tilde{d}_k^{(i)}$. $\tag{14}$

The modified inverse Daubechies D4 transform is specified by (for all $k = 0, 1, \ldots,$ $2^{n-i} - 1$ and $i \in \{1, 2, \ldots, n\}$):

1. $\tilde{d}_k^{(i)} = \dfrac{\sqrt{3} - 1}{\sqrt{2}} d_k^{(i)}, \qquad \tilde{s}_k^{(i)} = \dfrac{\sqrt{3} + 1}{\sqrt{2}} s_k^{(i)} - \tilde{d}_k^{(i)}.$

2. $s_{2k}^{(i-1)} = \tilde{s}_k^{(i)} - \sqrt{3} s_{2k+1}^{(i-1)}, \qquad s_{2k+1}^{(i-1)} = \dfrac{\sqrt{3}}{4} \tilde{s}_k^{(i)} - \dfrac{2 - \sqrt{3}}{4} \tilde{s}_{k-1}^{(i)} - \tilde{d}_{k-1}^{(i)}, \qquad (15)$

where $\tilde{s}_{k-1}^{(i)} := \tilde{s}_{k+l_i-1}^{(i)}, \tilde{d}_{k-1}^{(i)} := \tilde{d}_{k+l_i-1}^{(i)}$, for $k \in \{0, l_i, 2l_i, \ldots, (q_i - 1)l_i\}$.

Obviously, the modified DWT spectrum of a two-dimensional digital image X of size $N_1 \times N_2$ ($N_i = 2^{n_i}$, $n_i \in \mathrm{N}$, $i = 1, 2$) is obtained by repeated use of the one-dimensional modified DWT, i.e. N_2 times along the first spatial axis and then N_1 times along the second spatial axis.

3.2. *Computing the DWT Spectra of the Selected Signal (Image) Blocks*

To exhibit the extended potential capabilities of the modified DWT (Section 3.1), here a new generalized approach is proposed for developing fast computational algorithms to find the DWT spectra of the selected signal or image blocks (regions of interest – ROI). The proposed approach refers to the assumption that the modified DWT spectrum of the original signal (image) is known.

3.2.1. *One-Dimensional Case*
To begin with, we briefly investigate the situation concerning the one-dimensional case. Let

$$Y = \left(s_0^{(n)} d_0^{(n)} d_0^{(n-1)} d_1^{(n-1)} d_0^{(n-2)} d_1^{(n-2)} d_2^{(n-2)} d_3^{(n-2)} \ldots d_0^{(1)} d_1^{(1)} \ldots d_{N/2-1}^{(1)}\right)^T$$

be the modified DWT spectrum of the digital signal (data vector) $X = (x_0 x_1 x_2 \ldots x_{N-1})^T$, $N = 2^n$, $n \in \mathrm{N}$. Each decorrelated wavelet coefficient $d_j^{(i)}$ ($i \in \{p, p+1, \ldots, n\}$, $j \in \{0, 1, \ldots, 2^{n-i} - 1\}$) can evidently be put into a one-to-one correspondence with the signal block $X_j^{(i)} = (x_{2^i \cdot j} x_{2^i \cdot j+1} \ldots x_{2^i(j+1)-1})^T$, as the numerical value of $d_j^{(i)}$ is specified uniquely by $X_j^{(i)}$.

Let us denote the DWT spectrum of the signal block $X_j^{(i)}$ by $Y_j^{(i)}$. Obviously, $X_0^{(n)} = X$ and $Y_0^{(n)} = Y$.

To find $Y_j^{(i)}$, the following generalized algorithmic steps should be realized:

1. The very first spectral coefficient, i.e. the smoothed value $s_j^{(i)}$ in $Y_j^{(i)}$ should be computed using a newly developed generalized scheme (Fig. 1), where $j_0 = j$, $j_r = \lfloor j_{r-1}/2 \rfloor$, for all $r = 1, 2, \ldots, n - i$; here $\lfloor x \rfloor$ stands for the integral part of the real number x.

2. The remaining spectral coefficients in $Y_j^{(i)}$ should be extracted from the modified DWT spectrum Y of X, i.e. they should be identified with the ordered set of wavelet
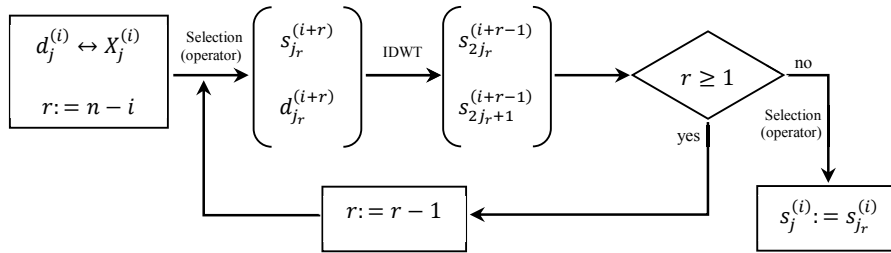
Fig. 1. The generalized scheme for finding the smoothed value $s_j^{(i)}$ in the DWT spectrum $Y_j^{(i)}$ of the signal block $X_j^{(i)}$.

coefficients

$$\{d_j^{(i)}, d_{2j}^{(i-1)}, d_{2j+1}^{(i-1)}, d_{4j}^{(i-2)}, d_{4j+1}^{(i-2)}, d_{4j+2}^{(i-2)}, d_{4j+3}^{(i-2)}, d_{8j}^{(i-3)}, \ldots, d_{8j+7}^{(i-3)}, \ldots,$$
$$d_{2^{i-1} \cdot j}^{(1)}, d_{2^{i-1} \cdot j+1}^{(1)}, \ldots, d_{2^{i-1}(j+1)-1}^{(1)}\}.$$

Thus, the DWT spectrum $Y_j^{(i)}$ of the signal block $X_j^{(i)}$ of a size not less than 2^p is computed and takes the form:

$$Y_j^{(i)} = \left(s_j^{(i)}\ d_j^{(i)}\ d_{2j}^{(i-1)}\ d_{2j+1}^{(i-1)}\ d_{4j}^{(i-2)}\ \ldots\ d_{2^{i-1} \cdot j}^{(1)}\ d_{2^{i-1} \cdot j+1}^{(1)}\ \ldots\ d_{2^{i-1}(j+1)-1}^{(1)}\right)^T. \quad (16)$$

Below, some particular cases are presented.

In the case of the discrete Haar transform (HT; Haar wavelets are fully localized in space), application of the generalized scheme (Fig. 1) leads to the following result:

$$s_j^{(i)} = \frac{1}{\sqrt{2^{n-i}}} s_0^{(n)} + \sum_{r=1}^{n-i} \frac{(-1)^{j_{r-1}}}{\sqrt{2^r}} \cdot d_{j_r}^{(i+r)}, \quad (17)$$

for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{0, 1, \ldots, 2^{n-i} - 1\}$.

In the case of the discrete Le Gall wavelet transform (DLGT), we have:

$$s_j^{(i)} = s_0^{(n)} - \frac{1}{2} \sum_{r=1}^{n-i} (-1)^{j_{r-1}} \cdot d_{j_r}^{(i+r)}, \quad (18)$$

for all $i \in \{p, p+1, \ldots, n\}$ and $j \in \{0, 1, \ldots, 2^{n-i} - 1\}$.

In the case of the discrete Daubechies D4 transform (D4):

$$s_j^{(i)} = \frac{1}{\sqrt{2^{n-i}}} s_0^{(n)} - \sum_{r=1}^{n-i} \frac{(-1)^{j_{r-1}}}{\sqrt{2^r}} \cdot d_{j_r}^{(i+r)}, \quad (19)$$

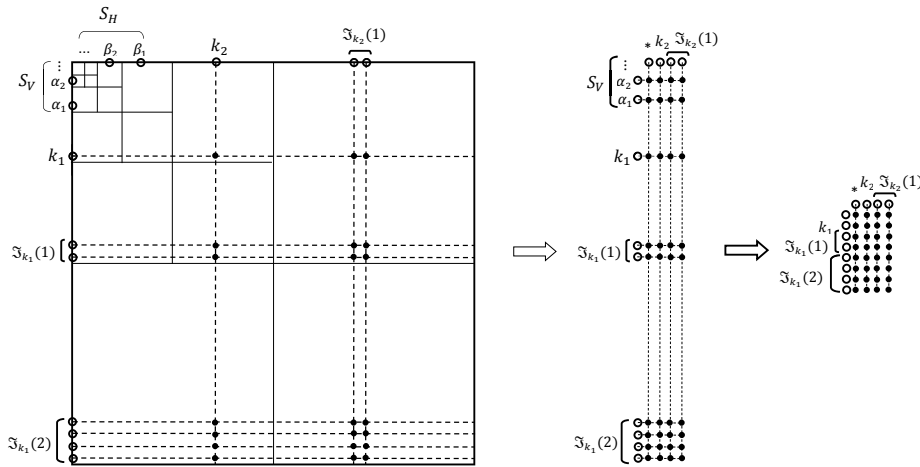for all $i \in \{p, p+1, \ldots, n\}$ and $j \in \{0, 1, \ldots, 2^{n-i} - 1\}$.

Fig. 2. The generalized scheme for computing the (modified) DWT spectrum $Y^{(k_1,k_2)}$ of the selected image block $X^{(k_1,k_2)}$.

Consistent patterns of the above relationships (expressions (17)–(19)), as well as knowledge of the detailed scheme for the direct evaluation of DWT spectra for data vectors (expressions (3), (4), (6) and (8), make it possible to compare both approaches (direct evaluation, proposed procedure), i.e. to estimate time expenditures associated with them. Comparative analysis results are presented in Section 4.

3.2.2. *Two-Dimensional Case*

Now, let $X = [X(m_1, m_2)]$ be a two-dimensional grey-level image of size $N \times N$ ($N = 2^n$, $n \in \mathbb{N}$) and let $Y = [Y(k_1, k_2)]$ be its two-dimensional modified (full decorrelation of wavelet coefficients across the scales not higher than p, $1 \leqslant p < n$) DWT spectrum.

Consider a wavelet coefficient $Y(k_1, k_2)$, $k_1, k_2 \in \{1, 2, \ldots, 2^{n-p+1} - 1\}$. Evidently, indices k_1 and k_2 can be presented in the form: $k_1 = 2^{n-i_1} + j_1$, $k_2 = 2^{n-i_2} + j_2$, where $i_1, i_2 \in \{1, 2, \ldots, n\}$, $j_1 \in \{0, 1, \ldots, 2^{n-i_1} - 1\}$, $j_2 \in \{0, 1, \ldots, 2^{n-i_2} - 1\}$.

It can be proved that the spectral coefficient $Y(k_1, k_2)$ is associated with the image block $X^{(k_1,k_2)} = [X(\tilde{m}_1, \tilde{m}_2)]$, where $(\tilde{m}_1, \tilde{m}_2) \in V_{k_1} \times V_{k_2}$ and $V_{k_r} = \{j_r 2^{i_r}, j_r 2^{i_r} + 1, \ldots, (j_r + 1)2^{i_r} - 1\}$, $r = 1, 2$. In other words, the numerical value of $Y(k_1, k_2)$ is uniquely specified by $X^{(k_1,k_2)}$.

Let us denote the (modified) DWT spectrum of the image block $X^{(k_1,k_2)}$ by $Y^{(k_1,k_2)} = [Y^{(k_1,k_2)}(u_1, u_2)]$, where $u_r \in \{0, 1, \ldots, 2^{i_r} - 1\}$, $r = 1, 2$.

Below, a new generalized scheme is proposed for finding the two-dimensional (modified) DWT spectrum $Y^{(k_1,k_2)}$ of the image block $X^{(k_1,k_2)}$ (Fig. 2).

In Fig. 2, the very first step is associated with repeated application of the generalized scheme (Fig. 1) to the selected rows of the DWT spectrum Y of X, namely, to the rows with serial numbers $k_1^* \in \{0\} \cup S_V \cup \{k_1\} \cup \Im_{k_1}$ (notation is clarified in the algorithm below). If we were interested in the image block $X^{(1,k_2)}$, then the scheme (Fig. 1) would have been applied to all rows of the DWT spectrum Y. The second step (Fig. 2) completes evaluation of the DWT spectrum $Y^{(k_1,k_2)}$ of the image block $X^{(k_1,k_2)}$, i.e. it applies the

same generalized scheme (Fig. 1) to the selected columns of the intermediate data array, i.e. to the columns with serial numbers $k_2^* \in \{*\} \cup \{k_2\} \cup \Im_{k_2}$.

So, we emphasize it once again, derivation of precise algorithmic formulae for finding the DWT spectra of the selected image blocks should match the main ideas presented in Fig. 2. Based on this understanding, the following algorithm has been developed.

Algorithm

1. Whatever the DWT, the following sets are formed:

$$S_V = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-i_1}\}, \quad \alpha_0 = k_1, \ \alpha_s = [\alpha_{s-1}/2], \ s = 1, 2, \ldots, n - i_1,$$

$$S_H = \{\beta_0, \beta_1, \ldots, \beta_{n-i_2}\}, \quad \beta_0 = k_2, \ \beta_t = [\beta_{t-1}/2], \ t = 1, 2, \ldots, n - i_2,$$

$$\Im_{k_r} = \{k_r\} \cup \left\{ \bigcup_{q=1}^{i_r-1} \Im_{k_r}(q) \right\},$$

$$\Im_{k_r}(q) = \left\{ 2^q \cdot k_r, 2^q \cdot k_r + 1, \ldots, 2^q \cdot (k_r + 1) - 1 \right\}, \quad r = 1, 2.$$

2. Now, the wavelet coefficients $Y^{(k_1,k_2)}(0, 0)$, $Y^{(k_1,k_2)}(u_1, 0)$ $(u_1 = 1, 2, \ldots, 2^{i_1} - 1)$ and $Y^{(k_1,k_2)}(0, u_2)$ $(u_2 = 1, 2, \ldots, 2^{i_2} - 1)$ of the image block $X^{(k_1,k_2)}$ are found in accordance with the formulae given below, namely:

   (a) In the case of Harr wavelet transform (HT),

$$Y^{(k_1,k_2)}(0, 0) = \frac{1}{\sqrt{2^{2n-i_1-i_2}}} Y(0, 0) + \frac{1}{\sqrt{2^{n-i_2}}} \sum_{s=1}^{n-i_1} \frac{(-1)^{\alpha_{s-1}}}{\sqrt{2^s}} \cdot Y(\alpha_s, 0)$$

$$+ \frac{1}{\sqrt{2^{n-i_1}}} \sum_{t=1}^{n-i_2} \frac{(-1)^{\beta_{t-1}}}{\sqrt{2^t}} \cdot Y(0, \beta_t)$$

$$+ \sum_{s=1}^{n-i_1} \sum_{t=1}^{n-i_2} \frac{(-1)^{\alpha_{s-1}+\beta_{t-1}}}{\sqrt{2^{s+t}}} \cdot Y(\alpha_s, \beta_t), \tag{20}$$

$$Y^{(k_1,k_2)}(u_1, 0) = \frac{1}{\sqrt{2^{n-i_2}}} Y(k_1^*, 0) + \sum_{t=1}^{n-i_2} \frac{(-1)^{\beta_{t-1}}}{\sqrt{2^t}} \cdot Y(k_1^*, \beta_t), \tag{21}$$

$$Y^{(k_1,k_2)}(0, u_2) = \frac{1}{2^{n-i_1}} Y(0, k_2^*) - \frac{1}{2} \sum_{s=1}^{n-i_1} \frac{(-1)^{\alpha_{s-1}}}{2^s} \cdot Y(\alpha_s, k_2^*), \tag{22}$$

   for all $u_1 = 1, 2, \ldots, 2^{i_1} - 1$ and $u_2 = 1, 2, \ldots, 2^{i_2} - 1$; k_1^* and k_2^* are the u_1-th and the u_2-th elements of the sets $\Im_{k_1}$ and $\Im_{k_2}$, respectively (numbering of elements in $\Im_{k_1}$ and $\Im_{k_2}$ starts with one);

(b) In the case of Le Gall wavelet transform (DLGT),

$$Y^{(k_1,k_2)}(0,0) = Y(0,0) - \frac{1}{2}\sum_{s=1}^{n-i_1}(-1)^{\alpha_s-1} \cdot Y(\alpha_s,0)$$

$$- \frac{1}{2}\sum_{t=1}^{n-i_2}(-1)^{\beta_t-1} \cdot Y(0,\beta_t)$$

$$+ \frac{1}{4}\sum_{s=1}^{n-i_1}\sum_{t=1}^{n-i_2}(-1)^{\alpha_s-1+\beta_t-1} \cdot Y(\alpha_s,\beta_t), \tag{23}$$

$$Y^{(k_1,k_2)}(u_1,0) = Y(k_1^*,0) - \frac{1}{2}\sum_{t=1}^{n-i_2}(-1)^{\beta_t-1} \cdot Y(k_1^*,\beta_t), \tag{24}$$

$$Y^{(k_1,k_2)}(0,u_2) = Y(0,k_2^*) - \frac{1}{2}\sum_{s=1}^{n-i_1}(-1)^{\alpha_s-1} \cdot Y(\alpha_s,k_2^*), \tag{25}$$

for all $u_1 = 1, 2, \ldots, 2^{i_1} - 1$ and $u_2 = 1, 2, \ldots, 2^{i_2} - 1$; k_1^* and k_2^* are the u_1-th and the u_2-th elements of the sets $\Im_{k_1}$ and $\Im_{k_2}$, respectively.

(c) In the case of Daubechies D4 wavelet transform, the wavelet coefficients of the image block $X^{(k_1,k_2)}$ are found using the expressions:

$$Y^{(k_1,k_2)}(0,0) = \frac{1}{\sqrt{2^{2n-i_1-i_2}}}Y(0,0) + \frac{1}{\sqrt{2^{n-i_2}}}\sum_{s=1}^{n-i_1}\frac{(-1)^{\alpha_s-1}}{\sqrt{2^s}} \cdot Y(\alpha_s,0)$$

$$+ \frac{1}{\sqrt{2^{n-i_1}}}\sum_{t=1}^{n-i_2}\frac{(-1)^{\beta_t-1}}{\sqrt{2^t}} \cdot Y(0,\beta_t)$$

$$+ \sum_{s=1}^{n-i_1}\sum_{t=1}^{n-i_2}\frac{(-1)^{\alpha_s-1+\beta_t-1}}{\sqrt{2^{s+t}}} \cdot Y(\alpha_s,\beta_t), \tag{26}$$

$$Y^{(k_1,k_2)}(u_1,0) = \frac{1}{\sqrt{2^{n-i_2}}}Y(k_1^*,0) + \sum_{t=1}^{n-i_1}\frac{(-1)^{\beta_t-1}}{\sqrt{2^t}} \cdot Y(k_1^*,\beta_t), \tag{27}$$

$$Y^{(k_1,k_2)}(0,u_2) = \frac{1}{\sqrt{2^{n-i_1}}}Y(0,k_2^*) + \sum_{s=1}^{n-i_1}\frac{(-1)^{\alpha_s-1}}{\sqrt{2^s}} \cdot Y(\alpha_s,k_2^*), \tag{28}$$

for all $u_1 = 1, 2, \ldots, 2^{i_1} - 1$ and $u_2 = 1, 2, \ldots, 2^{i_2} - 1$; k_1^* and k_2^* are the u_2-th and the u_2-th elements of the sets $\Im_{k_1}$ and $\Im_{k_2}$, respectively.

3. Whatever the DWT, the remaining spectral coefficients are derived directly from the DWT spectrum $Y = [Y(k_1, k_2)]$ of X, i.e.

$$Y^{(k_1,k_2)}(u_1,u_2) = Y(k_1^*,k_2^*), \tag{29}$$

for all $u_1 = 1, 2, \ldots, 2^{i_1} - 1$ and $u_2 = 1, 2, \ldots, 2^{i_2} - 1$; k_1^* and k_2^* are the u_2-th and the u_2-th elements of the sets $\Im_{k_1}$ and $\Im_{k_2}$, respectively.

4. The end. The two-dimensional modified DWT spectrum of the image block $X^{(k_1, k_2)}$ is computed, i.e. $Y^{(k_1, k_2)} = [Y^{(k_1, k_2)}(u_1, u_2)]$, where $u_1 \in \{0, 1, \ldots, 2^{i_1} - 1\}$ and $u_2 \in \{0, 1, \ldots, 2^{i_2} - 1\}$.

The efficiency of the above computational algorithms (expressions (20)–(29) for finding the DWT spectra of the selected image blocks (ROI) of size $2^p \times 2^p$ ($8 \leqslant p \leqslant n - 1$, $n = \log_2 N$; $N \times N$ being the size of the original image X), in comparison with direct evaluation of DWT spectra for the same blocks (Lifting Scheme; Section 2), is discussed in Section 4.

4. Experimental Analysis Results

First and foremost, the efficiency of the newly developed algorithms for finding the DWT spectra of the selected signal (image) blocks (Section 3.2) was analysed. In the one-dimensional case, the analysis was done for signal blocks of size 2^p ($8 \leqslant p \leqslant n - 1$; $N = 2^n$ being the size of the signal X). The achievable speed gains (Table 1) were expressed in terms of $\rho = \tau_d / \tau_{pr}$, where τ_d specifies the time needed for direct evaluation of DWT spectra for the indicated signal blocks (Section 2) and τ_{pr} – for the time needed by the proposed algorithmic steps (Section 3.2.1).

As it can be seen (Table 1), the performance of the developed approach to finding the DWT spectra of the selected signal blocks (ROI) depends directly on the size of ROI, i.e. the greater the size of ROI, the higher comparative performance is achieved.

In the two-dimensional case, the achievable impressive speed gains, expressed in terms of $\rho = \tau_d / \tau_{pr}$, where τ_d specifies the time needed for direct evaluation of DWT spectra for the indicated image blocks (Section 2) and τ_{pr} – for the time needed by the proposed algorithm (Section 3.2.2), are presented in Table 2.

Undoubtedly, fast passage from the modified DWT spectrum of the digital signal (image) under processing to DWT spectra of the selected signal (image) blocks (ROI; Table 1 and Table 2) opens the door to many new real-time applications of the modified version of DWT, for instance, signal (image) feature extraction in the wavelet domain, signal (image) pattern recognition, locally progressive signal (image) encoding, the detection and localization of defects in textural images, etc. Some interesting applications of the modified DLGT for locally progressive encoding of ROI in a digital signal (image), where the

Table 1

A comparison of two approaches to finding the DWT spectra for the selected signal blocks of size 2^p.

N	p											
	8			9			10			11		
	HT	DLGT	D4	HT	DLGT	D4	HT	DLGT	D4	HT	DLGT	D4
1024	5.25	87.1	13.2	5.56	122	27.1	–	–	–	–	–	–
2048	4.81	76.3	12.8	4.94	127	26.5	5.76	160	54.3	–	–	–
4096	3.33	76.3	12.6	5.22	122	25.7	5.67	172	53.0	5.50	238	108.5

Table 2
A comparison of the two approaches to finding the DWT spectra of the selected image blocks of size $2^p \times 2^p$.

$N \times N$	p								
	8			9			10		
	HT	DLGT	D4	HT	DLGT	D4	HT	DLGT	D4
512×512	29.8	526	379	–	–	–	–	–	–
1024×1024	19.5	505	281	21.9	678	401	–	–	–
2048×2048	12.0	489	255	17.2	648	375	20.1	629	383

achievable speed gains appear to be significant, are presented in Valantinas (2009) and Kančelkis and Valantinas (2012).

On the other hand, the digital data processing (especially, compression) performance is closely related to the energy compaction property of the transform, i.e. how the energy of the transform is distributed among its coefficients (Hsu and Wu, 1997; Singh and Kumar, 2010). The latter distribution is commonly characterized by the rate of downtrend of the transform (spectral) coefficients, as their serial numbers increase. The faster the downtrend of spectral coefficients, the better the energy compaction property is prescribed to the transform.

To evaluate the downtrend of spectral coefficients of a particular discrete transform, the hyperbolic image filters can be used. The hyperbolic image filtering idea is itself rather simple (Zinterhof and Zinterhof, 1993). Let $X = [X(m_1, m_2)$ $(m_1, m_2 \in \{0, 1, \ldots, N - 1\})$ be a two-dimensional digital image and let $Y = [Y(k_1, k_2)]$ be its two-dimensional discrete spectrum. Now, if M $(1 \leqslant M < (N - 1)^2)$ is a priori chosen integer, then only those spectral coefficients $Y(k_1, k_2)$ whose serial numbers k_1 and k_2 satisfy the condition $k_1 \cdot k_2 \leqslant M$ should be stored. When reconstructing the image (obtaining its estimate $\tilde{X}$), the remaining spectral coefficients $Y(k_1, k_2)$ $(k_1 \cdot k_2 > M)$ are set to zero, i.e. the hyperbolic image filtering idea utilizes the supposition that the human eye is less sensitive to changes in higher frequencies than in lower ones.

Thus, the higher the quality of the restored image $\tilde{X}$ (the image compression ratio being fixed), the better energy distribution is incidental to the discrete transform in question.

In this paper, to evaluate tentatively the energy compaction property of the modified DWT, in comparison with its basic version, hyperbolic filtering was applied to the standard grey level image *Lena.bmp* of size 256×256 (Fig. 3(a); *USC SIPI Image Database*), with a view to highlight and compare the quality of restored images (image estimates), in the mean squared error

$$\delta = \delta(X, \tilde{X}) = \left(\frac{1}{N^2} \sum_{m_1, m_2 = 0}^{N-1} \left(X(m_1, m_2) - \tilde{X}(m_1, m_2) \right)^2 \right)^{1/2}$$

sense (Table 3).

As it can be seen (Table 3), in the case of DLGT the quality of restored images remains, practically, unchanged provided decorrelation of wavelet coefficients is made at the scale $p \in \{6, 7\}$. These facts are worthy of note when applying a particular modified DWT to
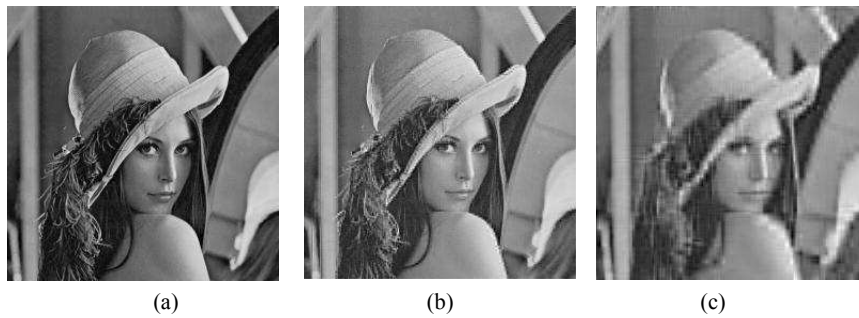
(a)   (b)   (c)

Fig. 3. Test image *Lena.bmp* and some of its estimates; (a) original image 256×256; (b) image estimate ($\beta = 5$, $\delta = 9.1$; DLGT); (c) image estimate ($\beta = 10$, $\delta = 14.2$; D4).

Table 3

Analysis of the energy compaction property of the modified DWT, based on hyperbolic filtering of the digital image *Lena.bmp*, 256×256.

Image compression ratio, β	DWT		The modified version of DWT							
			$p = 4$		$p = 5$		$p = 6$		$p = 7$	
	DLGT	D4	DLGT	D4	DLGT	D4	DLGT	D4	DLGT	D4
2	4.2	6.6	4.6	12.5	4.4	17.3	4.3	19.7	4.2	6.7
5	9.1	11.3	11.2	15.0	10.2	19.2	9.4	21.5	9.2	11.5
10	13.1	14.2	16.6	16.9	15.4	20.6	14.1	23.0	13.4	14.8
15	15.3	16.8	19.5	18.7	18.1	21.8	16.5	21.8	15.7	17.0

digital image (signal) compression. In other real-time applications, such as image feature extraction, localization of defects in textural images, etc., difficulties of the type described are not encountered.

5. Conclusion

In this paper, a new modified version of the discrete wavelet transform (DWT) is proposed. This new version of DWT is characterized by apparently improved space localization properties of the transform, i.e. by significantly better decorrelation of wavelet coefficients across the (lower) scales. The developed generalized ideas can be applied to any DWT of higher order (Le Gall, integer-to-integer Le Gall, Daubechies D4, CDF 9/7, etc.).

This proposed modified version of DWT has made it possible to develop an original algorithm for extremely fast computation of the DWT spectra for smaller image blocks (ROI, selected at the user's request). The procedure applies the improved space localization properties of the modified DWT and relies on the fact that the modified DWT spectrum of the original image is known.

We strongly believe that the proposednew developments will find various applications in implementing efficient and up-to-date digital data processing technologies in areas such

as image (signal) pattern recognition, image (signal) feature extraction, locally progressive image (signal) encoding associated with transmitting graphical data across a low communications channel (say, Internet), localization of defects in textural images and so on. In other words, we openly believe that the proposed novel ideas will capture the attention of research in the field.

In the short term, similar research concerning the reversible Le Gall wavelet transform (facilitating lossless compression of digital images), as well as CDF 9/7 transform, is supposed. In parallel, some real-time applications, associated with locally progressive image encoding and analysis of textural images, are planned.

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Diskrečiųjų bangelių transformacijų lokalizavimo erdvėje savybių gerinimas

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Šiame straipsnyje siūloma nauja modifikuota diskrečiosios bangelių transformacijos (DBT) versija, pasižyminti akivaizdžiai geresnėmis lokalizavimo erdvėje savybėmis bei ženkliai padidėjusiomis potencinėmis praktinio panaudojimo galimybėmis. Pasiūlymo (požiūrio) esmė – užtikrinama pilnoji spektrinių DBT koeficientų dekoreliacija aukštuose dažniuose. Pateiktas požiūris gali būti pritaikytas bet kuriai aukštesnės eilės DBT (Le Gall, Daubechies D4, CDF 9/7 ir pan.). Siekiant pademonstruoti atsivėrusias naujas potencines modifikuotos DBT praktinio panaudojimo galimybes, sudaryta originali išskirtinai greita DBT spektro apskaičiavimo pasirinktiems signalo (vaizdo) blokams procedūra. Kartu straipsnyje pateikiami ir aptariami eksperimento, skirto modifikuotos DBT energijos pakavimo savybėms tirti, rezultatai.