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Resolving Rank Reversal in TOPSIS: A Comprehensive Analysis of Distance Metrics and Normalization Methods

Huan-Jyh SHYUR[1,*], Hsu-Shih SHIH[2]

[1] *Department of Information Management, Tamkang University, Taiwan*
[2] *Department of Management Sciences, Tamkang University, Taiwan*
e-mail: 110620@mail.tku.edu.tw, hshih@mail.tku.edu.tw

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Abstract. This paper examines ranking reversal (RR) in Multiple Criteria Decision Making (MCDM) using the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS). Through a mathematical analysis of min-max and max normalization techniques and distance metrics (Euclidean, Manhattan, and Chebyshev), the study explores their impact on RR, particularly when new, high-performing alternatives are introduced. This research provides insight into the causes of RR, offering a framework that clarifies when and why RR occurs. The findings help decision-makers select appropriate techniques, promoting more consistent and reliable outcomes in real-world MCDM applications.

Key words: ranking reversal, TOPSIS, normalization, distance metric, extreme alternative.

1. Introduction

Multiple-Criteria Decision Making (MCDM) is a systematic approach used to make decisions in complex scenarios where multiple conflicting criteria or objectives need to be considered simultaneously. In many real-world situations, decisions cannot be made solely based on a single criterion; instead, multiple factors with different levels of importance and often incommensurable units need to be taken into account. MCDM provides a structured framework to evaluate and compare different alternatives in a way that reflects the preferences and priorities of decision makers.

During the MCDM process, the data representing different criteria might come in various formats, units, and scales. These criteria could include quantitative measurements like costs, benefits, or performance metrics, as well as qualitative factors like risk levels or expert opinions. Due to these differences, directly comparing and combining these criteria for decision-making purposes can be challenging and lead to misleading results. To deal with the problem, normalization has become a fundamental process in MCDM that enables decision makers to effectively compare and evaluate alternatives across multiple

*Corresponding Author

criteria (Pavličić, 2001). It generally supports the goal of making well-informed, objective, and consistent decisions that accurately reflect the preferences and priorities of decision makers.

Distance-based models in MCDM play a crucial role in quantifying the dissimilarity between alternatives, and the issue of rank reversal (RR) has drawn significant attention within this context. RR denotes a phenomenon where the rank order of alternatives undergoes a change due to alterations in the evaluated alternatives or related parameters. This phenomenon contradicts our intuitive expectations and is formally termed the axiom of independence of irrelevant alternatives (IIA) in the multi-attribute utility theory (von Winterfeldt and Edwards, 1986). Belton and Gear (1983) initially identified RR in the Analytic Hierarchy Process (AHP), attributing it to AHP's relative measurement mode. Salo and Hämäläinen (1997) extended this finding, linking RR to AHP's mode of measurement. The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), applied to MCDM, was later reported to exhibit RR by Triantaphyllou (2000). Subsequent debates in major operations research journals, including works by Saaty (1990a, 1990b, 1991), Saaty and Vargas (1984), Triantaphyllou (2001), and Kulakowski *et al.* (2019), further underscored the significance of RR.

TOPSIS stands out as a straightforward and intuitive method that accommodates both the positive and negative aspects of each alternative, making it a widely adopted approach in decision-making processes dealing with multiple, conflicting criteria. It has been still well received until recently, e.g. Biswas *et al.* (2024) and Kousar *et al.* (2024). Despite its popularity, TOPSIS is not immune to RR, although its occurrence is notably less pronounced compared to AHP (Zanakis *et al.*, 1998). The RR phenomenon in TOPSIS arises from the sensitivity of the distance metric to variations in the attribute values of alternatives. The normalization process in TOPSIS involves dividing the performance values by their respective weights. Furthermore, the choice of distance metrics and the determination of the positive-ideal solution (PIS) and negative-ideal or anti-negative solution (NIS) can introduce subjectivity, influencing the final rankings. Therefore, careful consideration of these factors is crucial to mitigate the RR problem and enhance the robustness of the TOPSIS method in decision-making processes.

To address the issue of RR, previous researchers such as Senouci *et al.* (2016) and Yang (2020) have suggested approaches like linear normalization and using absolute positive-ideal and negative-ideal solutions. However, many studies have relied on simplistic, fabricated examples or simulations to illustrate the RR phenomenon, leaving its real-world impact and significance unclear (Shih and Olson, 2022). This research is motivated by the need for a more comprehensive understanding of RR in the TOPSIS method, driving an in-depth mathematical analysis of the underlying formulae to better quantify and explain the occurrence of RR.

Distance metrics, such as Manhattan, Euclidean, and Chebyshev, measure the dissimilarity or proximity between alternatives in different ways, each introducing variations in rankings due to their sensitivity to the scale and distribution of data. These variations can trigger RR by affecting the computed distances between alternatives. To mitigate RR, it is crucial to examine how these distance metrics influence the final rankings, providing insight into the underlying mechanics of RR. This study aims to fill a gap in the literature by exploring the role of these distance measures in the TOPSIS framework, offering a more detailed understanding of their influence on ranking reversals.

While the Chebyshev distance in TOPSIS offers robust ranking results, it has limitations in capturing subtle performance differences across multiple criteria. This lack of sensitivity to nuanced variations can lead to less precise distinctions between alternatives. Additionally, although Euclidean distance has been shown to outperform Manhattan distance in reducing RR, it may not fully eliminate the phenomenon. The study supports these conclusions through a case example, providing empirical validation for the effectiveness of different distance metrics in preventing RR, while acknowledging the limitations inherent to each approach.

The rest of the paper runs as follows. Section 2 reviews the existing resolutions for RR in TOPSIS. Section 3 delves into an examination of the ranking index of TOPSIS, presenting a mathematical perspective on TOPSIS using different distance metric methods for handling RR. Section 4 examines a case study. Section 5 offers concluding remarks on RR solutions for TOPSIS.

2. Literature Review

RR can occur in an MCDM process when a new alternative is introduced or an existing one is removed from the candidate list, all without any adjustments to the criteria weights. This phenomenon is rooted in the intricate interactions between alternatives and their relative positions in the ranking. With the addition of a new alternative, the dynamic relationships among all options might shift, causing some alternatives to be reevaluated in the context of the changed landscape. Similarly, the removal of an alternative can lead to redistributions in the ranking as the absence of that option could amplify the importance of others. Rank reversal serves as a reminder of the delicate balance within decision frameworks, highlights the non-intuitive nature of multi-criteria scenarios, and underscores the need for a thoughtful and adaptable approach to decision-making.

Building on the observations of the RR phenomenon in TOPSIS, Ren *et al.* (2007) presented pioneering research in addressing RR within TOPSIS. They introduced a novel modified synthetic evaluation method wherein two separation measures are transformed into a two-dimensional plane, and the ranking index is determined by the distance to PIS minus the minimum of all alternatives to PIS. However, it appears that their proposal lacks a systematic approach to avoiding the RR phenomenon, and the examples presented, such as those in the public health system and student evaluation, serve primarily for illustrative purposes.

Wang and Luo (2009) contributed to the literature by showcasing an example highlighting the occurrence of RR in the TOPSIS algorithm. However, they did not specifically identify the impact arising from non-dominated alternatives.

Zavadskas *et al.* (2006) studied the impact of vector and max-min normalization on TOPSIS ranking accuracy. Chakraborty and Yeh (2009) simulated comparisons of one vector and three linear normalization methods for TOPSIS, advocating for vector normalization based on ranking consistency and weight sensitivity. Çelen (2014) compared four normalization methods from Chakraborty and Yeh (2009) in evaluating the Turkish deposit banking market, affirming vector normalization's reliability and noting that max-min and max normalization also produced consistent results. Kong (2011) aimed to mitigate RR by introducing linear normalization and incorporating extreme fictitious alternatives to stabilize PIS and NIS. However, Kong's example was limited to five alternatives with two criteria.

García-Cascale and Lamata (2012) similarly considered the approaches introduced by Kong (2011). They further elaborated on numerous situations, presenting an example with four alternatives and two criteria.

İç (2014) explored the integration of design of experiments (DoE) into TOPSIS for the financial performance evaluation of companies. The study concluded with a five-alternative case involving five criteria to illustrate the DoE-TOPSIS method, aiming to preserve ranking. However, despite the belief that this combination could avoid RR, the method did not demonstrate improvement in the steps of TOPSIS.

Cables *et al.* (2016) proposed a reference ideal method, redefining the distance to the reference ideal as an interval and linearly normalizing performance data into a value. They also modified the separation measures, making the reference ideal as the vector of weights. The method was illustrated using a personnel selection case with five candidates and six criteria, claiming independence from the type of data. However, verification of this illustration is challenging due to the assumed interval values and reliance on a single example.

Senouci *et al.* (2016) suggested a modified TOPSIS for selecting a mobile network interface, incorporating linear normalization with four types and dynamically setting max/min values against RR. Their case involved seven alternatives and was evaluated by five criteria through simulation. Despite achieving a zero RR ratio with two types of normalization, we will later demonstrate that their case is unable to avert the RR phenomenon.

In addition to the developments above, Kuo (2017) investigated weighting on the elements of the TOPSIS ranking index, showing that the proposed modified ranking index outperforms the traditional TOPSIS in ranking consistency. Mufazzal and Muzakkir (2018) proposed a proximity index to overcome RR, utilizing Manhattan distance to count separation measures and using the sum of distances as the ranking index. Although they provided seven historical cases as evidence for the proposed approach, they were unable to clarify the ranking differences observed in their outcomes or elucidate the properties of the index.

Authors de Farias Aires and Ferreira (2019) introduced two forms of linear normalization to modify the traditional TOPSIS. They demonstrated the robustness of their approach through a student selection case involving twenty alternatives with three criteria and their subsets.

Yang (2020) made modifications to the approach proposed by de Farias Aires and Ferreira (2019), incorporating linear normalization of historical extreme values and assigning weights only to PIS. The study used Wang and Luo's questionable example (Wang and Luo, 2009) to showcase the rank stability achieved by their approach.

Table 1
Classification of proposed solutions for mitigating RR in TOPSIS.

Category	Author(s) (year)
I	Li (2009); Kong (2011); García-Cascale and Lamata (2012); Cables *et al.* (2016); Senouci *et al.* (2016); de Farias Aires and Ferreira (2019); Yang (2020); Tiwari and Kumar (2021); Yang *et al.* (2022); Ciardiello and Genovese (2023)
II	Kong (2011); García-Cascale and Lamata (2012); İç (2014); Cables *et al.* (2016); Senouci *et al.* (2016); de Farias Aires and Ferreira (2019); Yang (2020); Tiwari and Kumar (2021); Yang *et al.* (2022)
III	Mufazzal and Muzakkir (2018)
IV	Ren *et al.* (2007); Li (2009); Kuo (2017); Mufazzal and Muzakkir (2018); Yang *et al.* (2022)
V	Ren *et al.* (2007); İç (2014); Shen (2021)

Note: Some studies have proposed solutions involving more than one category.

Based on the studies mentioned above, the proposed solutions to mitigate RR are classified into the following five main categories.

- Category I: Alteration of normalization formulae, which include approaches such as linear normalization that deviate from the traditional TOPSIS method.
- Category II: Modification of the determination of PIS and/or NIS, departing from the conventional approach of maximizing and minimizing performance values on each criterion of all existing alternatives.
- Category III: Change in the calculation formulae of separation measures, aiming to refine the metrics used to assess the distance between alternatives and the ideal solutions.
- Category IV: Reconstruction of the components of relative closeness, which involves redefining the parameters used to evaluate the proximity of alternatives to the ideal solutions.
- Category V: Deviation from or addition of extra steps to the traditional TOPSIS procedure, introducing new methodologies or steps to mitigate the occurrence of RR and enhance the robustness of the decision-making process.

It is observed that the five categories mentioned above are arranged according to the steps of the TOPSIS procedure. Table 1 summarizes other research efforts, and the proposed strategies aimed at alleviating RR in TOPSIS. The majority of proposed solutions are concentrated in Categories I and II, while fewer solutions are found in the other categories. Categories I, II, and III primarily revolve around the selection of suitable normalization methods and modifications to the original separation measure formulae. The choice of a distance measurements plays a crucial role in assessing the relative proximity of alternatives to PIS and NIS, aside from modifying the determination of PIS and/or NIS. However, there is a lack of literature that explicitly address the relationship between RR and the specific method used for distance measures. As a result, our focus is directed towards exploring the effects of different normalization methods and distance metrics, particularly the Minkowski metric.

3. A Mathematical Exploration of Normalization and Distance Measurements in TOPSIS

This section provides a comprehensive mathematical explanation of how normalization and distance measurements contribute to the phenomenon of RR. This investigation aims to understand the factors that give rise to RR in MCDM, with a particular focus on TOPSIS. The goal is to enhance decision-making by addressing the occurrences of rank reversal, which are comparatively less frequent in TOPSIS. The common algorithm of TOPSIS for ranking and selection includes the following seven steps (Hwang and Yoon, 1981).

- **Step 1:** Create a decision or evaluation matrix $D = \{x_{ij}\}$. The matrix refers to m alternatives and n criteria, with its performance element x_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$.
- **Step 2:** Construct the normalized decision matrix $R = \{r_{ij}\}$. Matrix D is normalized to matrix R with

$$r_{ij} = \phi(x_{ij}), \quad i = 1, \dots, m, \; j = 1, \dots, n, \tag{1}$$

  ϕ represents a specific type of normalization operation.
- **Step 3:** Construct the weighted normalized decision matrix V. The matrix $V = \{v_{ij}\} = \{w_j r_{ij}\}$ is calculated by multiplying the elements at each column of the matrix R by their associated weights w_j, $j = 1, \dots, n$.
- **Step 4:** Determine the positive-ideal and negative-ideal solutions V^+ (PIS) and V^- (NIS), respectively.

$$\begin{aligned} V^+ &= [v_1^+, \dots, v_n^+] = \Big[\Big(\max_i v_{ij} \big| j \in J\Big), \Big(\min_i v_{ij} \big| j \in J^*\Big)\Big], \\ V^- &= [v_1^-, \dots, v_n^-] = \Big[\Big(\min_i v_{ij} \big| j \in J\Big), \Big(\max_i v_{ij} \big| j \in J^*\Big)\Big]. \end{aligned} \tag{2}$$

  J is associated with the benefit criteria (the properties whose higher values are desirable), and J^* is associated with the cost criteria.
- **Step 5:** For each alternative $i = 1, \dots, m$, calculate the separation measures S_i^+ and S_i^-. The separation measures can be quantified using the n-dimensional distance function, denoted as d:

$$S_i^+ = d\big(\{v_{ij}\}, V^+\big), \qquad S_i^- = d\big(\{v_{ij}\}, V^-\big). \tag{3}$$

- **Step 6:** For each alternative $i = 1, \dots, m$, calculate its relative closeness C_i^*:

$$C_i^* = S_i^- / \big(S_i^+ + S_i^-\big), \tag{4}$$

  where $0 \leqslant C_i^* \leqslant 1$. The larger the index value is, the better is the performance of the alternative.
- **Step 7:** Rank the preference order of all alternatives with the descending order of the value of C_i^*.

In the TOPSIS methodology, when comparing two alternatives, A_p and A_q, their respective relative closeness values are calculated as follows: $C_p^* = S_p^- / (S_p^+ + S_p^-)$ for alternative A_p and $C_q^* = S_q^- / (S_q^+ + S_q^-)$ for alternative A_q. If alternative A_p is preferred over alternative A_q in TOPSIS, then C_p^* must be greater than C_q^*. This can be expressed as:

$$S_p^- / (S_p^+ + S_p^-) > S_q^- / (S_q^+ + S_q^-). \tag{5}$$

After transposing the terms, the inequation can be rewritten as:

$$S_p^- S_q^+ - S_p^+ S_q^- > 0. \tag{6}$$

The mentioned expression is the fundamental formula employed to determine the preference relationship between alternatives A_p and A_q in TOPSIS. By assessing the result of $S_p^- S_q^+ - S_p^+ S_q^-$, it is possible to establish whether A_p is preferable to A_q. A positive value indicates that A_p is preferred over A_q.

The mathematical models are based on a scenario in which a new alternative with the highest performance in a specific criterion, referred to as criterion k, is added to the candidate list. All criteria are considered as benefit criteria. Benefit criteria represent the positive attributes or qualities that decision-makers are trying to optimize or maximize in their decision-making process.

Similar to many MCDM methods, TOPSIS utilizes criteria weights in its aggregation process. These weights, assigned to criteria, play a crucial role in measuring the overall preferences of alternatives. Different sets of weights can significantly impact the ranks of alternatives (Kaliszewski *et al.*, 2018). However, this study solely focuses on the distance measurements and normalization methods, thereby disregarding the weighting effect. Hence, all criteria weights are assumed to be identical and unchanged. Consequently, we assume $w_j = 1/n$ for all criteria j. Figure 1 outlines the scope of this mathematical evaluation. The forthcoming exploration will delve into the mathematical aspects to identify the fundamental causes of RR in TOPSIS, particularly focusing on the impact of the normalization process and the use of different distance metrics. Initially, the max-min normalization method will be employed as a basis for analysis. The study will examine how different distance metrics—namely Euclidean, Manhattan, and Chebyshev—affect the ranking of alternatives, driving a detailed mathematical investigation into the underlying formulae to better quantify and explain the occurrence of RR. In the subsequent section, the analysis will extend to explore the effects of the max normalization method.

3.1. *Max-Min Normalization*

The normalization process is designed to allow fair comparison of alternatives across diverse criteria. It is a crucial step in the MCDM process, because it helps ensure fair and meaningful comparisons among different criteria or alternatives. Pena *et al.* (2022) explored normalization techniques for both quantitative and qualitative criteria in MCDM.

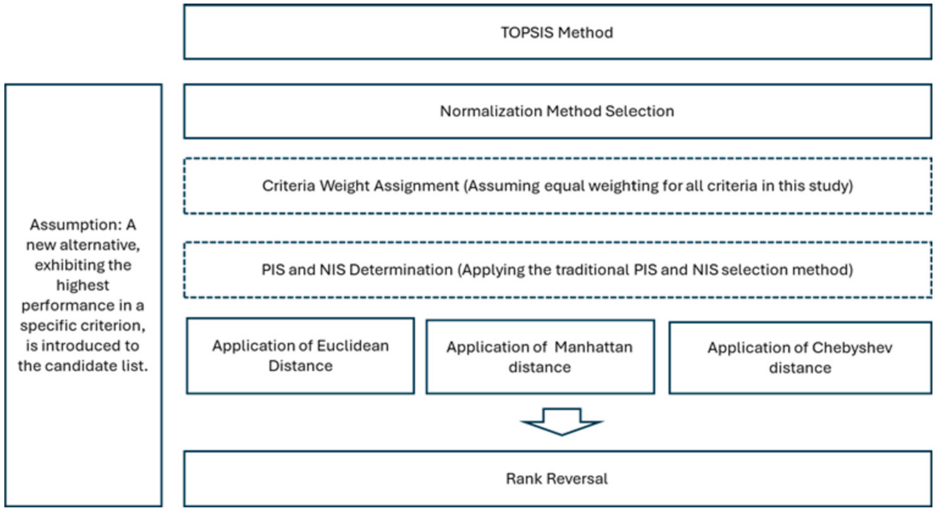

Fig. 1. Scope of the study.

They particularly focused on normalizing qualitative data using fuzzy concepts. However, when a new alternative is added, its performance on existing criteria can disrupt the normalization balance. If the new alternative performs exceptionally well on certain criteria compared to the existing alternatives, then the new alternative is disproportionately favoured in the ranking, possibly leading to a reversal in the original ranking order.

Chen (2019) investigated the influence of three normalization methods (sum, max-min, and vector) on entropy-based TOPSIS. Results indicated normalization notably impacted outcomes by shaping criteria data diversity. Chen advised against normalization in entropy weight computation and suggested vector normalization for TOPSIS. This study conducts max-min normalization. Among the normalization methods discussed, the max-min normalization method exhibits a certain behaviour. When a new alternative is introduced, only the normalized values associated with the specified *kth* criterion are adjusted, while the normalized values of all original alternatives under other criteria remain unchanged. This simplifies the complexity of our further analysis on the impact of distance measures on rank reversal. Before the new alternative with its performance element $x_{new,j}$, $j = 1, \dots, n$ is added, the normalized performance r_{ij} of alternative i on criterion j can be calculated as:

$$r_{ij} = (x_{ij} - \min x_{.j})/(\max x_{.j} - \min x_{.j}), \tag{7}$$

where $\max x_{.j}$ and $\min x_{.j}$ are the maximum and minimum values of all alternatives' performances on criterion j, respectively.

Attention is now directed to criterion k, where the new alternative exhibits the highest performance. Upon the introduction of the new alternative, its normalized performance $r_{new,k}$ is 1, as it possesses the highest performance. At the same time, the normalized performances r_{ik} of other alternatives might be influenced by the new range of values. Let $\Delta_k = x_{new,k}/\max x_{.k}$, and the original r_{ik} is transformed into:

$$r'_{ik} = \frac{x_{ik} - \min x_{.k}}{\max x_{.k}\Delta_k - \min x_{.k}} = \delta_k r_{ik}, \tag{8}$$

where $0 \leqslant \delta_k = (\max x_{.k} - \min x_{.k})/(\max x_{.k} - \min x_{.k}) \leqslant 1$. Since the new normalized values in criterion k are computed by multiplying the original normalized values by δ_k, the successive calculation in TOPSIS to determine the final ranking significantly depends on the value of δ_k. It is important to note that if max normalization is employed as a replacement for max-min normalization, then parameter δ_k can be adjusted to $1/\Delta_k$.

Since it is already established that $\Delta_k > 1$, it follows that $r'_{ik} < r_{ik}$. Accordingly, due to the normalization process, all other alternatives, excluding the new one, experience decreased scores. However, the positive-ideal and negative-ideal solutions, denoted as V^+ (PIS) and V^- (NIS), respectively, remain unchanged, given that $V^+ = [v_1^+, \dots, v_n^+] = [1/n, \dots, 1/n]$ and $V^- = [v_1^-, \dots, v_n^-] = [0, \dots, 0]$.

The next step is to explore how different distance measurements impact the ranking results when using max-min normalization scores. This study specifically considers three distance measures used in TOPSIS: Euclidean, Manhattan, and Chebyshev distances.

3.2. *Euclidean Distance*

The Euclidean distance is a straightforward and widely used measurement that calculates the straight-line distance between two points in a multi-dimensional space. In this case, following the TOPSIS method and utilizing Euclidean distance for separation measures, the initial two separation measures, S_i^+ and S_i^-, before introducing the new alternative are as follows:

$$S_i^+ = \frac{1}{n}\sqrt{\sum_{j=1}^{n}(r_{ij} - 1)^2}, \qquad S_i^- = \frac{1}{n}\sqrt{\sum_{j=1}^{n}(r_{ij})^2}. \tag{9}$$

Within the candidate list, select any two alternatives, A_p and A_q, where A_p is preferred over A_q according to the TOPSIS method. It can then be observed that the expression $S_p^- S_q^+ - S_p^+ S_q^- > 0$, or equivalently, $S_p^- S_q^+ > S_p^+ S_q^-$, holds true. Representing this expression using the Euclidean metric, the following result is derived:

$$\sqrt{\sum_{j=1}^{n}(r_{pj})^2} \cdot \sqrt{\sum_{j=1}^{n}(r_{qj} - 1)^2} > \sqrt{\sum_{j=1}^{n}(r_{pj} - 1)^2} \cdot \sqrt{\sum_{j=1}^{n}(r_{qj})^2}. \tag{10}$$

However, upon introducing the new alternative in this case, the separation measures are modified to:

$$S_i^{+'} = \frac{1}{n}\sqrt{\sum_{j=1}^{n}(r_{ij}-)^2 + \left(r'_{ik} - 1\right)^2 - (r_{ik}-1)^2},$$
$$S_i^{-'} = \frac{1}{n}\sqrt{\sum_{j=1}^{n} r_{ij}^2 + r_{ik}'^2 - r_{ik}^2}. \tag{11}$$

Given that alternative A_p is initially preferred to A_q according to the TOPSIS method, when the new alternative is introduced to the list of candidates, it is imperative to uphold the following formula to avoid alterations in the ranking order.

$$S_p^{-'}S_q^{+'} > S_p^{+'}S_q^{-'} \quad \text{or} \quad S_p^{-'}S_q^{+'} - S_p^{+'}S_q^{-'} > 0. \tag{12}$$

In mathematical terms, this relationship is expressed as:

$$\sqrt{\sum_{j=1}^{n}\left((r_{pj})^2 - r_{pk}^2\left(1-\delta_k^2\right)\right)}\sqrt{\sum_{j=1}^{n}\left((r_{qj}-1)^2 + (\delta_k r_{qk}-1)^2 - (r_{qk}-1)^2\right)}$$
$$> \sqrt{\sum_{j=1}^{n}\left((r_{pj}-1)^2 + (\delta_k r_{pk}-1)^2 - (r_{pk}-1)^2\right)}\sqrt{\sum_{j=1}^{n}\left((r_{qj})^2 - r_{qk}^2\left(1-\delta_k^2\right)\right)}, \tag{13}$$

where it is observed that $r_{ik}^2(1-\delta_k^2) > 0$ and $(\delta_k r_{ik}-1)^2-(r_{ik}-1)^2 > 0$. To comprehend how the introduction of a new alternative, excelling in criterion k, to the candidate list influences the likelihood of RR, two distinct scenarios covering various conditions are explored.

Scenario 1: $x_{pk} > x_{qk}$. In the given scenario, it is apparent that $r_{pk} > r_{qk}$ and $r'_{pk} > r'_{qk}$. Furthermore, the following inequalities are valid:

$$r_{pk}^2\left(1-\delta_k^2\right) > r_{qk}^2\left(1-\delta_k^2\right) > 0. \tag{14}$$

Moreover, it is ensured that if $r_{pk} + r_{qk} < 2/(\delta_k + 1)$, then

$$(\delta_k r_{pk} - 1)^2 - (r_{pk} - 1)^2 > (\delta_k r_{qk} - 1)^2 - (r_{qk} - 1)^2 > 0; \tag{15}$$

otherwise,

$$(\delta_k r_{qk} - 1)^2 - (r_{qk} - 1)^2 > (\delta_k r_{pk} - 1)^2 - (r_{pk} - 1)^2 > 0. \tag{16}$$

It is noted that $r_{pk} + r_{qk} < 1$ and $1 \leqslant 2(\delta_k + 1) \leqslant 2$. Therefore, $r_{pk} + r_{qk} < 2(\delta_k + 1)$, which implies that $(\delta_k r_{pk} - 1)^2 - (r_{pk} - 1)^2$ is always larger than $(\delta_k r_{qk} - 1)^2 - (r_{qk} - 1)^2$.

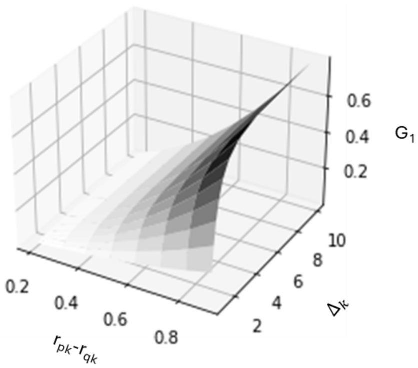

Fig. 2. The gap G_1 as a function of r_{pk} and Δ_k.

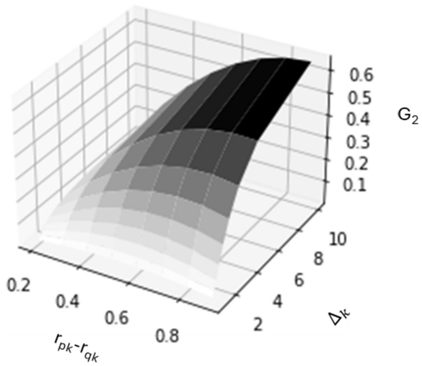

Fig. 3. The gap G_2 as a function of r_{pk} and Δ_k.

The sum of squared values $\sum_{j=1}^{n}(r_{ij})^2$ in essence significantly surpasses $r_{ik}^2 - r_{ik}'^2$, and $\sum_{j=1}^{n}(r_{ij}-1)^2$ is notably larger than $(\delta_k r_{ik}-1)^2-(r_{ik}-1)^2$. In most cases, the condition $S_p^{-\prime} S_q^{+\prime} > S_p^{+\prime} S_q^{-\prime}$ can be maintained when the new alternative is introduced. However, if $S_p^- S_q^+ \cong S_q^- S_p^+$, then the final ranking between alternatives p and q can be altered especially when $r_{pk}^2(1-\delta_k^2) >> r_{qk}^2(1-\delta_k^2)$ and $(\delta_k r_{pk}-1)^2-(r_{pk}-1)^2 \gg (\delta_k q_{ik}-1)^2-(r_{qk}-1)^2$. Let G_1 represent the gap between $r_{pk}^2(1-\delta_k^2)$ and $r_{qk}^2(1-\delta_k^2)$, and G_2 represent the gap between $(\delta_k r_{pk}-1)^2-(r_{pk}-1)^2$ and $(\delta_k r_{qk}-1)^2-(r_{qk}-1)^2$. It is obvious that Δ_k amplifies the difference between $r_{pk}^2(1-\delta_k^2)$ and $r_{qk}^2(1-\delta_k^2)$, leading to an increased gap. Similarly, the disparity between $(\delta_k r_{pk}-1)^2-(r_{pk}-1)^2$ and $(\delta_k r_{qk}-1)^2-(r_{qk}-1)^2$ also widens. Figures 2 and 3 illustrate the values of G_1 and G_2 for varying $r_{pk}-r_{qk}$ and Δ_k, with r_{qk} is fixed at 0.1. The parameter Δ_k plays a critical role in amplifying both gap measures. The impact is particularly strong at higher $r_{pk}-r_{qk}$ values. For example, when $r_{pk}=0.9$, increasing Δ_k from 1 to 10 results in a substantial growth in both G_1 and G_2. At lower $r_{pk}-r_{qk}$, the effect of Δ_k is less pronounced but still present.

Scenario 2: $x_{pk} \leqslant x_{qk}$. Given $r_{qk} > r_{pk}, r'_{qk} > r'_{pk}$, it can be demonstrated that:

$$\begin{aligned} & r_{qk}^2\left(1-\delta_k^2\right) > r_{pk}^2\left(1-\delta_k^2\right) > 0 \quad \text{and} \\ & (\delta_k r_{qk}-1)^2-(r_{qk}-1)^2 > (\delta_k r_{pk}-1)^2-(r_{pk}-1)^2 > 0. \end{aligned} \tag{17}$$

As a result, the inequality $S_p^{-\prime} S_q^{+\prime} > S_p^{+\prime} S_q^{-\prime}$ still upholds. The possibility of RR occurring is negligible.

3.3. *Manhattan Distance*

Manhattan distance, also known as city block distance, calculates the sum of the absolute differences between the coordinates of two points. When applied to determine the relative closeness of alternatives to PIS and NIS in this case, the initial separation measures for S_i^+ and S_i^- are as follows:

$$S_i^+ = \frac{1}{n}\sum_{j=1}^{n}(1-r_{ij}), \qquad S_i^- = \frac{1}{n}\sum_{j=1}^{n} r_{ij}. \tag{18}$$

Assuming that A_p is initially preferred over A_q as per the TOPSIS method, a result arises: $S_p^- S_q^+ > S_p^+ S_q^-$. This inequality holds significance and can be expressed more explicitly as:

$$\sum_{j=1}^{n} r_{pj} \sum_{j=1}^{n}(1-r_{qj}) > \sum_{j=1}^{n}(1-r_{pj}) \sum_{j=1}^{n} r_{qj}. \tag{19}$$

However, upon the introduction of the new alternative, the separation measures undergo adjustments:

$$\begin{aligned} S_i^{+\prime} &= \frac{1}{n}\left(\sum_{j=1}^{n}(1-r_{ij}) + r_{ik}(1-\delta_k)\right), \\ S_i^{-\prime} &= \frac{1}{n}\left(\sum_{j=1}^{n} r_{ij} - r_{ik}(1-\delta_k)\right). \end{aligned} \tag{20}$$

To maintain the existing ranking order when introducing a new alternative, the following formula must be satisfied:

$$S_p^{-\prime} S_q^{+\prime} > S_p^{+\prime} S_q^{-\prime} \quad \text{or} \quad S_p^{-\prime} S_q^{+\prime} - S_p^{+\prime} S_q^{-\prime} > 0. \tag{21}$$

In mathematical terms, this relationship can be expressed as:

$$\left(\sum_{j=1}^{n} r_{pj} - r_{pk}(1-\delta_k)\right)\left(\sum_{j=1}^{n}(1-r_{qj}) + r_{qk}(1-\delta_k)\right)$$
$$> \left(\sum_{j=1}^{n}(1-r_{pj}) + r_{pk}(1-\delta_k)\right)\left(\sum_{j=1}^{n} r_{qj} - r_{qk}(1-\delta_k)\right). \tag{22}$$

If $x_{pk} > x_{qk}$, then it follows that $r_{pk} > r_{qk}$. In cases where $S_p^- S_q^+ \cong S_q^- S_p^+$, the outcomes of $r_{pk}(1-\delta_k)$ and $r_{qk}(1-\delta_k)$ can have a notable impact on the final ranking between alternatives p and q. The final ranking between alternatives p and q can be altered when $r_{pk}(1-\delta_k) \gg r_{qk}(1-\delta_k)$. This can occur when $r_{pk} > r_{qk}$ and Δ_k is a substantially larger value (with δ_k representing a small value). It is worth noting that the impact of changes in δ_k in Eq. (22) is more pronounced compared to Eq. (13). This observation underscores the fact that employing Euclidean distance can be more effective in reducing the occurrence of the RR scenario when compared to using Manhattan distance.

3.4. *Chebyshev Distance*

Chebyshev distance calculates the maximum absolute difference between corresponding attributes. It is suitable for scenarios where only the most extreme differences are of interest. When using Chebyshev distance, the initial separation measures for S_i^+ and S_i^- are computed as follows:

$$S_i^+ = \max_j \frac{1}{n}(1-r_{ij}), \qquad S_i^- = \max_j \frac{1}{n} r_{ij}. \tag{23}$$

After introduction of the new alternative, the updated separation measures are given by:

$$\begin{aligned} S_i^{+\prime} &= \max\left\{\frac{1}{n}(1-r_{i1}), \frac{1}{n}(1-r_{i2}), \dots, \frac{1}{n}(1-\delta_k r_{ik}), \dots, \frac{1}{n}(1-r_{im})\right\}, \\ S_i^{-\prime} &= \max\left\{\frac{1}{n}r_{i1}, \frac{1}{n}r_{i2}, \dots, \frac{1}{n}\delta_k r_{ik}, \dots, \frac{1}{n}r_{im}\right\}. \end{aligned} \tag{24}$$

If the maximum value does not exist in the kth criterion before and after the introduction of the new alternative, then the separation measures remain unchanged, and no RR occurs; otherwise, assuming $x_{pk} > x_{qk}$, the following six extreme conditions should be explored to assess the potential occurrence of the RR phenomenon. These conditions are detailed in Table 2. The symbol "*" denotes the separation measures with the maximum value in the kth criterion. Table 3 illustrates the outcomes, indicating whether $S_p^{-\prime} S_q^{+\prime}$ is greater than $S_p^{+\prime} S_q^{-\prime}$ or not, given the condition $S_p^- S_q^+ > S_p^+ S_q^-$, where l, m, and j represent criteria other than the kth criterion.

As δ_k decreases, $\frac{1}{n}(1-\delta_k r_{pk})$ and $\frac{1}{n}(1-\delta_k r_{qk})$ increase, while $\frac{1}{n}\delta_k r_{pk}$ and $\frac{1}{n}\delta_k r_{qk}$ decrease. The significance of δ_k in determining the inequality is evident. Under conditions

Table 2
Conditions for assessing RR phenomenon under various scenarios.

Conditions	S_p^-	S_p^+	S_q^-	S_q^+	$S_p^{-\prime}$	$S_p^{+\prime}$	$S_q^{-\prime}$	$S_q^{+\prime}$
1	*		*		*		*	
2		*		*		*		*
3	*				*			
4			*				*	
5		*				*		
6				*				*

Table 3
Assessment of RR conditions considering different scenarios.

Conditions	$S_p^{-\prime} S_q^{+\prime}$	$S_p^{+\prime} S_q^{-\prime}$	RR status
1	$\frac{1}{n^2}\delta_k r_{pk}(1-r_{ql})$	$\frac{1}{n^2}\delta_k r_{qk}(1-r_{pm})$	No
2	$\frac{1}{n^2} r_{pm}(1-\delta_k r_{qk})$	$\frac{1}{n^2} r_{ql}(1-\delta_k r_{pk})$	Potential
3	$\frac{1}{n^2}\delta_k r_{pk}(1-r_{ql})$	$\frac{1}{n^2} r_{ql}(1-r_{pm})$	Potential
4	$\frac{1}{n^2} r_{pm}(1-r_{ql})$	$\frac{1}{n^2}\delta_k r_{qk}(1-r_{pj})$	No
5	$\frac{1}{n^2} r_{pm}(1-r_{ql})$	$\frac{1}{n^2} r_{ql}(1-\delta_k r_{pk})$	Potential
6	$\frac{1}{n^2} r_{pm}(1-\delta_k r_{qk})$	$\frac{1}{n^2} r_{ql}(1-r_{pj})$	No

1, 4, and 6, it is readily demonstrated that $S_p^{-\prime} S_q^{+\prime}$ remains larger than $S_p^{+\prime} S_q^{-\prime}$, precluding the occurrence of RR. However, under conditions 2, 3, and 5, the possibility of RR arises. Condition 2 states that both alternatives A_p and A_q exhibit the maximum distance from the PIS value in the kth criterion, both before and after the introduction of the new alternative. In Condition 3, alternative A_p is specified to maintain the maximum distance from the NIS value in the kth criterion, both before and after the introduction of the new alternative. Condition 5 underscores the focus on alternative A_p, consistently revealing the longest distance to the PIS value in the kth criterion, both preceding and succeeding the introduction of the new alternative. It is crucial to note that RR is not a common outcome in the majority of conditions.

4. Illustrative Case

Hwang and Yoon's fighter selection problem is modified to consider only the first three benefit criteria for interpretation (Hwang and Yoon, 1981). Table 4 shows the decision matrix. Criterion X_1 corresponds to maximum speed, X_2 pertains to ferry range, and X_3 relates to maximum payload. Table 5 shows the normalized decision matrix using the min-max normalization method. Utilizing the TOPSIS method, Tables 6, 7, and 8 showcases the ranking outcomes obtained by employing Manhattan distance, Euclidean distance, and Chebyshev distance to calculate separation measures, respectively, without taking criteria weights into account.

Table 4
Decision matrix of the four alternatives evaluated by three criteria.

Alternatives	Criteria		
	X_1	X_2	X_3
A_1	2.0	1.500	20.000
A_2	2.5	2.700	18.000
A_3	1.8	2.000	21.000
A_4	2.2	1.800	20.000

Table 5
Normalized decision matrix.

Alternatives	Criteria		
	X_1	X_2	X_3
A_1	0.2857	0	0.6667
A_2	1	1	0
A_3	0	0.4167	1
A_4	0.5714	0.25	0.6667

Table 6
Results of ranking based on Manhattan distance.

Alternatives	Separation measures			Rank
	S^+	S^-	C^*	
A_1	2.0476	0.9524	0.3175	4
A_2	1	2	0.6667	1
A_3	1.5833	1.4167	0.4722	3
A_4	1.5119	1.4881	0.4960	2

Table 7
Results of ranking based on Euclidean distance.

Alternatives	Separation measures			Rank
	S^+	S^-	C^*	
A_1	1.2733	0.7253	0.3629	4
A_2	1	1.4142	0.5858	1
A_3	1.1577	1.0833	0.4834	3
A_4	0.9259	0.9129	0.4965	2

Table 8
Results of ranking based on Chebyshev distance.

Alternatives	Separation measures			Rank
	S^+	S^-	C^*	
A_1	1	0.6667	0.4000	4
A_2	1	1	0.5000	1
A_3	1	1	0.5000	1
A_4	0.7500	0.6667	0.4706	3

According to Tables 6 and 7, using Manhattan and Euclidean distances yields identical ranking results. The initial ranking result shows $A_2 \succ A_4 \succ A_3 \succ A_1$. However, when employing Chebyshev distance, the separation measures calculate the distance between the alternative and PIS/NIS, considering only the maximum absolute difference among the three criteria. Under such circumstances, the outcomes are determined solely by the performance of the alternative in a single criterion. Therefore, when an alternative excels in a particular criterion, it secures the optimal ranking. For example, A_3 outperforms in criterion X_3, resulting in a priority 1 ranking, or the same as A_2. Because using Chebyshev distance in calculating separation measures does not simultaneously consider the influence of all criteria, the resulting ranking may be biased. Evaluating the performance of alternatives neglects other criteria and focuses solely on one criterion with extreme values. Therefore, adopting Chebyshev distance to compute separation measures in most MCDM problems might not be entirely appropriate. It could be challenging to differentiate rankings in the performance assessment of many candidate solutions.

To simulate a scenario where RR might arise, alternative A5 is introduced for evaluation based on the criteria $[X_1 = 2, X_2 = 2,700, X_3 = 21,000]$. Despite the addition of A5, PIS remains unchanged. The new rankings, using Manhattan and Euclidean distances, are $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$., However, when employing Chebyshev distance, the rankings are $A_5 \succ A_2 = A_3 \succ A_4 \succ A_1$. Importantly, the order of A_1, A_2, A_3, and A_4 remains consistent by using the same distance measurement, indicating the absence of RR. The achievement rating of alternative A_5 on criterion X_2 is gradually increased, leading to a change in PIS. In the current scenario, when the achievement rating of alternative A_5 on criterion X_2 continues to increase, the ratio of change Δ_{X_2} is greater than 1 and increases synchronously. Tables 9, 10, and 11 illustrate the effect of increasing Δ_{X_2} on alternatives A_2 and A_4 by using various distance measurements.

The results indicate that, initially, A_2 is preferred over A_4 when $\Delta_{X_2} = 1$, regardless of the distance measurements employed. The preference order remains consistent with the situation before the introduction of A_5 to the candidate list. As Δ_{X_2} increases to 2.0, signifying a twofold improvement in A_5's performance in criterion X_2, RR occurs when Manhattan distance is employed to calculate separation measures (see Table 9). Interestingly, A_4 outperforms A_2 in this scenario, although A_2 is the superior solution before the introduction of A_5. If Euclidean distance is utilized, then RR emerges when Δ_{X_2} increases to 2.6 (see Table 10).

As discussed in Section 3, it is highlighted that Euclidean distance has the potential to decrease the likelihood of RR compared to using Manhattan distance. In this case, the results are consistent with the previously stated expectations. In Table 10, it is evident that A_2 consistently outperforms A_4, irrespective of the increasing values of Δ_{X_2}. This consistent superiority indicates the absence of RR. A_2 excels in X_1 and has the poorest performance in X_3, but not in X_2, while A_4 does not exhibit the best or worst performance in any criteria. Notably, A_4 has the longest distance to PIS in criterion X_2 both before and after the introduction of A_5. Additionally, Table 4 confirms that A_2 performs better than A_4 in criterion X_2, aligning with condition 6 in Table 11. Hence, using Chebyshev distance to calculate separation measures in this case ensures the absence of RR.

Table 9
Effect of the change ratio Δ_{X_2} in PIS on rank reversal via Manhattan distance.

Δ_{X_2}	$S_2^{+\prime}$	$S_2^{-\prime}$	C_2^*	$S_4^{+\prime}$	$S_4^{-\prime}$	C_4^*	Ranking
1.0	1.0000	2.0000	0.6667	1.5119	1.4881	0.4960	$A_2 \succ A_4$
1.1	1.1837	1.8163	0.6054	1.5578	1.4422	0.4807	$A_2 \succ A_4$
1.2	1.3103	1.6897	0.5632	1.5895	1.4105	0.4702	$A_2 \succ A_4$
1.3	1.4030	1.5970	0.5323	1.6127	1.3873	0.4624	$A_2 \succ A_4$
1.4	1.4737	1.5263	0.5088	1.6303	1.3697	0.4566	$A_2 \succ A_4$
1.5	1.5294	1.4706	0.4902	1.6443	1.3557	0.4519	$A_2 \succ A_4$
1.6	1.5745	1.4255	0.4752	1.6555	1.3445	0.4482	$A_2 \succ A_4$
1.7	1.6117	1.3883	0.4628	1.6648	1.3352	0.4451	$A_2 \succ A_4$
1.8	1.6429	1.3571	0.4524	1.6726	1.3274	0.4425	$A_2 \succ A_4$
1.9	1.6694	1.3306	0.4435	1.6793	1.3207	0.4402	$A_2 \succ A_4$
2.0	1.6923	1.3077	0.4359	1.6850	1.3150	0.4383	$A_4 \succ A_2$
2.1	1.7122	1.2878	0.4293	1.6900	1.3100	0.4367	$A_4 \succ A_2$
2.2	1.7297	1.2703	0.4234	1.6943	1.3057	0.4352	$A_4 \succ A_2$
2.3	1.7452	1.2548	0.4183	1.6982	1.3018	0.4339	$A_4 \succ A_2$
2.4	1.7590	1.2410	0.4137	1.7017	1.2983	0.4328	$A_4 \succ A_2$
2.5	1.7714	1.2286	0.4095	1.7048	1.2952	0.4317	$A_4 \succ A_2$
2.6	1.7826	1.2174	0.4058	1.7076	1.2924	0.4308	$A_4 \succ A_2$
2.7	1.7927	1.2073	0.4024	1.7101	1.2899	0.4300	$A_4 \succ A_2$
2.8	1.8020	1.1980	0.3993	1.7124	1.2876	0.4292	$A_4 \succ A_2$
2.9	1.8104	1.1896	0.3965	1.7145	1.2855	0.4285	$A_4 \succ A_2$
3.0	1.8182	1.1818	0.3939	1.7165	1.2835	0.4278	$A_4 \succ A_2$

Table 10
Effect of the change ratio Δ_{X_2} in PIS on rank reversal via Euclidean distance.

Δ_{X_2}	$S_2^{+\prime}$	$S_2^{-\prime}$	C_2^*	$S_4^{+\prime}$	$S_4^{-\prime}$	C_4^*	Ranking
1.0	1.0000	1.4142	0.5858	0.9259	0.9129	0.4965	$A_2 \succ A_4$
1.1	1.0167	1.2909	0.5594	0.9635	0.9015	0.4834	$A_2 \succ A_4$
1.2	1.0471	1.2148	0.5371	0.9898	0.8948	0.4748	$A_2 \succ A_4$
1.3	1.0781	1.1647	0.5193	1.0092	0.8906	0.4688	$A_2 \succ A_4$
1.4	1.1065	1.1300	0.5053	1.0242	0.8879	0.4644	$A_2 \succ A_4$
1.5	1.1315	1.1052	0.4941	1.0360	0.8859	0.4609	$A_2 \succ A_4$
1.5	1.1315	1.1052	0.4941	1.0360	0.8859	0.4609	$A_2 \succ A_4$
1.6	1.1533	1.0868	0.4852	1.0456	0.8845	0.4583	$A_2 \succ A_4$
1.7	1.1722	1.0728	0.4778	1.0536	0.8834	0.4561	$A_2 \succ A_4$
1.8	1.1888	1.0619	0.4718	1.0603	0.8826	0.4543	$A_2 \succ A_4$
1.9	1.2034	1.0532	0.4667	1.0660	0.8819	0.4528	$A_2 \succ A_4$
2.0	1.2163	1.0463	0.4624	1.0709	0.8814	0.4515	$A_2 \succ A_4$
2.1	1.2277	1.0406	0.4588	1.0752	0.8810	0.4504	$A_2 \succ A_4$
2.2	1.2379	1.0359	0.4556	1.0790	0.8806	0.4494	$A_2 \succ A_4$
2.3	1.2471	1.0319	0.4528	1.0823	0.8804	0.4485	$A_2 \succ A_4$
2.4	1.2554	1.0286	0.4503	1.0853	0.8801	0.4478	$A_2 \succ A_4$
2.5	1.2630	1.0258	0.4482	1.0880	0.8799	0.4471	$A_2 \succ A_4$
2.6	1.2698	1.0234	0.4463	1.0904	0.8797	0.4465	$A_4 \succ A_2$
2.7	1.2761	1.0213	0.4445	1.0926	0.8796	0.4460	$A_4 \succ A_2$
2.8	1.2819	1.0194	0.4430	1.0946	0.8794	0.4455	$A_4 \succ A_2$
2.9	1.2872	1.0178	0.4416	1.0965	0.8793	0.4451	$A_4 \succ A_2$
3.0	1.2921	1.0164	0.4403	1.0982	0.8792	0.4446	$A_4 \succ A_2$

Table 11
Effect of the change ratio Δ_{X_2} in PIS on rank reversal via Chebyshev distance.

Δ_{X_2}	$S_2^{+'}$	$S_2^{-'}$	C_2^*	$S_4^{+'}$	$S_4^{-'}$	C_4^*	Ranking
1.0	1.0000	1.0000	0.5000	0.7500	0.6667	0.4706	$A_2 \succ A_4$
1.1	1.0000	1.0000	0.5000	0.7959	0.6667	0.4558	$A_2 \succ A_4$
1.2	1.0000	1.0000	0.5000	0.8276	0.6667	0.4462	$A_2 \succ A_4$
1.3	1.0000	1.0000	0.5000	0.8507	0.6667	0.4393	$A_2 \succ A_4$
1.4	1.0000	1.0000	0.5000	0.8684	0.6667	0.4343	$A_2 \succ A_4$
1.5	1.0000	1.0000	0.5000	0.8824	0.6667	0.4304	$A_2 \succ A_4$
1.6	1.0000	1.0000	0.5000	0.8936	0.6667	0.4273	$A_2 \succ A_4$
1.7	1.0000	1.0000	0.5000	0.9029	0.6667	0.4247	$A_2 \succ A_4$
1.8	1.0000	1.0000	0.5000	0.9107	0.6667	0.4226	$A_2 \succ A_4$
1.9	1.0000	1.0000	0.5000	0.9174	0.6667	0.4209	$A_2 \succ A_4$
2.0	1.0000	1.0000	0.5000	0.9231	0.6667	0.4194	$A_2 \succ A_4$
2.1	1.0000	1.0000	0.5000	0.9281	0.6667	0.4180	$A_2 \succ A_4$
2.2	1.0000	1.0000	0.5000	0.9324	0.6667	0.4169	$A_2 \succ A_4$
2.3	1.0000	1.0000	0.5000	0.9363	0.6667	0.4159	$A_2 \succ A_4$
2.4	1.0000	1.0000	0.5000	0.9398	0.6667	0.4150	$A_2 \succ A_4$
2.5	1.0000	1.0000	0.5000	0.9429	0.6667	0.4142	$A_2 \succ A_4$
2.6	1.0000	1.0000	0.5000	0.9457	0.6667	0.4135	$A_2 \succ A_4$
2.7	1.0000	1.0000	0.5000	0.9482	0.6667	0.4128	$A_2 \succ A_4$
2.8	1.0000	1.0000	0.5000	0.9505	0.6667	0.4122	$A_2 \succ A_4$
2.9	1.0000	1.0000	0.5000	0.9526	0.6667	0.4117	$A_2 \succ A_4$
3.0	1.0000	1.0000	0.5000	0.9545	0.6667	0.4112	$A_2 \succ A_4$

Figure 4 illustrates the variations in the functions of $S_2^{-'}S_4^{+'} - S_2^{+'}S_4^{-'}$ in response to changes in Δ_{X_2} using different distance measurements. When the values of these functions decrease below 0, it indicates a shift in the preference order between A_2 and A_4, transitioning from $A_2 \succ A_4$ to $A_4 \succ A_2$. When utilizing Manhattan distance, it is evident that the decreasing slope of the function is steeper compared to using Euclidean distance. This suggests that using Manhattan distance can make it more susceptible to the occurrence of RR. If the primary concern is to avoid RR, then opting for Chebyshev distance can be the best solution.

For comparison, max normalization is now employed in this example, utilizing a fixed NIS in the process, denoted as $V^- = [v_1^-, \dots, v_n^-] = [0, \dots, 0]$. Figure 5 depicts the fluctuations in the functions of $S_2^{-'}S_4^{+'} - S_2^{+'}S_4^{-'}$ in response to variations in Δ_{X_2}, employing different distance measurements. It is evident that employing max normalization appears to yield a more resilient ranking result compared to using max-min normalization. No instances of RR occurred even after increasing Δ_{X_2} to 10. Notably, when utilizing Manhattan distance, the decreasing slope of the function is steeper compared to using Euclidean distance. However, the utilization of Chebyshev distance still yields highly robust results.

Max-min normalization scales the performance scores based on the range of values for each criterion. It is sensitive to outliers and can be influenced by extreme values. For instance, consider alternative A_5, which exhibits an extreme performance score in X_2 as Δ_{X_2} continues to increase. This situation emphasizes the sensitivity of max-min normalization to outliers, potentially leading to a more pronounced impact on the rank-

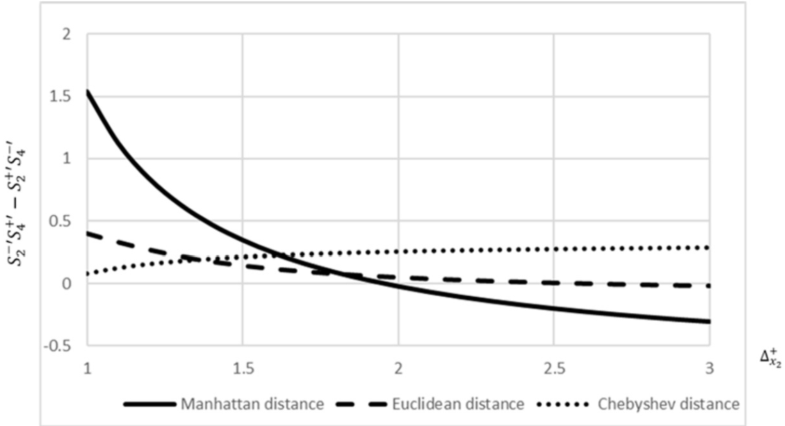

Fig. 4. Effect of the change ratio Δ_{X_2} on alternatives A_2 and A_4 via Max-Min normalization.

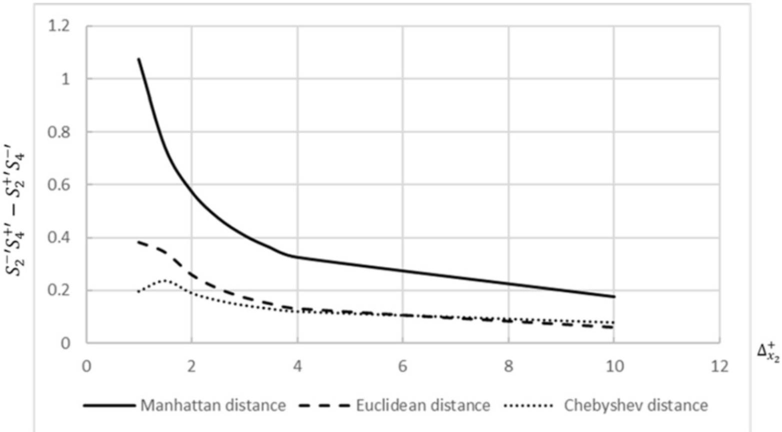

Fig. 5. Effect of the change ratio Δ_{X_2} on alternatives A_2 and A_4 via Max normalization.

ing outcomes, especially when certain alternatives demonstrate extreme performance in a particular criterion.

5. Conclusion and Remarks

This paper presents a detailed mathematical analysis of the RR phenomenon in the context of TOPSIS, with a focus on its distance metrics and normalization. The objective is to enhance the TOPSIS process by understanding and addressing the occurrence of RR.

The exploration focused on how the introduction of a new alternative excelling in a specific criterion could disrupt the normalization balance and potentially lead to RR. The study analysed the impact of min-max normalization scores on ranking outcomes using

Euclidean, Manhattan, and Chebyshev distance measures. Mathematical expressions were examined for separation measures before and after the introduction of a new alternative. Euclidean distance showed scenarios where RR might occur, particularly when the introduced alternative performed well in a specific criterion. Manhattan distance revealed similar trends, while Chebyshev distance highlighted conditions for RR under extreme differences. Overall, Euclidean distance appeared to outperform Manhattan in preventing RR.

Despite offering valuable insights into RR in MCDM processes within the TOPSIS framework, the study is limited to min-max and max normalization and three distance measures, with other normalization methods and dynamic criteria weights left unexplored. Managers should carefully select distance measures and normalization techniques based on the decision-making context, particularly when introducing strong alternatives could alter rankings. Future research could explore additional normalization techniques, distance metrics, dynamic criteria weighting, and hybrid approaches, providing further practical insights and enhancing the robustness of MCDM tools across industries.

In conclusion, the mathematical analysis provides valuable insights into the conditions and factors contributing to the phenomenon of RR in MCDM processes, specifically within the TOPSIS framework. By exploring different normalization methods and distance measures, we contribute to a deeper understanding of decision-making challenges and improve the robustness of MCDM methodologies in practical applications. The findings underscore the importance of careful consideration and customization of methods based on specific decision contexts to mitigate the risk of RR and to enhance the reliability of decision outcomes.

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H.J. Shyur received his PhD from the Department of Industrial and Systems Engineering at Rutgers University. After earning his doctorate, he worked at the Federal Aviation Administration (FAA) Technical Center in the United States, where he led the SPAS (Safety Performance Analysis System) project. He is currently a full-time professor in the Department of Information Management at Tamkang University. During his tenure at Tamkang University, Dr. Shyur actively participated in aviation safety projects with the Civil Aeronautics Administration (CAA) of Taiwan and contributed to the development of the FORAS (Flight Operations Risk Assessment System) project in collaboration with EVA Air. His research interests include aviation safety management, reliability engineering, decision theory, and the application of artificial intelligence.

H.S. Shih, emeritus professor of Management Sciences at Tamkang University, Taiwan, earned his PhD in Industrial and Manufacturing Systems Engineering from Kansas State University in 1995. He also holds an MS and BS from Chung Cheng Institute of Technology, Taiwan. Shih's research spans decision analysis, operations research, soft computing, and environmental management, with numerous publications in top journals like the *European Journal of Operational Research* and *Journal of Cleaner Production*. Recognized on Stanford/Elsevier's Top 2% Most Influential Scientists list (2022–2024), he has authored three books and four book chapters. Shih has served as president of the Operations Research Society of Taiwan and held leadership roles in the INFORMS Taiwan Chapter and the International Society on Multiple Criteria Decision Making. Additionally, he has been a visiting scholar at the University of Pittsburgh, Aalto University, and the Norwegian School of Economics.