# NEARLY NONSTATIONARY ARMA PROCESSES: SECOND ORDER PROPERTIES

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**Abstract.** Second order properties of nearly nonstationary ARMA processes are investigated in the cases when the autoregressive polynomial equation has (i) a real root close to 1; (ii) a real root close to –1; (iii) a pair of complex roots close to the unit circle.

The effect of the closeness to the unit circle of the ARMA poles on its covariance and spectral density functions is considered. The obtained results demonstrate three specific ways of degeneracy of these functions, as the roots tend to 1 in modulus. As a consequence three different estimates of the ARMA parameters located in the neighbourhood of the border of the stationarity region for ARMA process are derived and their asymptotic distributions are examined.

**Key words:** nearly nonstationary ARMA process; nearly unit roots; degeneration of covariance and spectral density functions.

**1. Introduction.** The focus of attention in time series analysis has recently been shifted from a short memory process $\{X_t, \ t \in Z\}$, such as represented by traditional ARMA $(p, q)$ models to a long memory process such as fractional ARIMA $(p, d, q)$, $(-0.5 < d < 0.5)$ or nearly nonstationary ARMA process. There is an evidence that such process occur frequently in hydrology, economics, high precision measurements. The theory of autoregressive (AR) time series with some zeroes of the AR polynomial on or close to the unit circle has been started as early as 1958 when White derived the limiting distribution of an estimated parameter $\hat{a}_1$ for AR(1) in both cases: $a_1 \neq 1$ and $a_1 = 1$ but then the unit root problem was neglected for many years. The interest renewed in 1976 when Fuller (1976) and Dickey and Fuller (1979, 1981) developed statistical tests for detecting the presence of a unit root in the AR(1) and general AR($p$) case.

In general case it is difficult to distinguish between a nonstationary process such as ARIMA($p, 1, q$) and an ARMA($p + 1, q$) process, whose autoregressive polynomials has zeroes close to the unit circle from a sample of finite length. If an estimated covariance function is a slowly decaying function, the numerical computation of parameter's estimates can be quite critical for some methods of estimation. For example, if we apply the Yule-Walker equations to fit an

ARMA$(p, q)$ model with poles close to the unit circle, the estimated covariance matrix $\hat{R}_p$ is nearly singular and a fitted model can bear little resemblance to the true model.

There are now relatively complete theories (Yap and Reinsel, 1995; Chan, 1990) for dealing with the inference for nearly nonstationary ARMA models and nonstationary autoregressive integrated moving average model. But the effect of closeness to the unit circle of any ARMA pole on the second order characteristics of an ARMA process has not been studied quantitatively, which is the main focal point of the present paper.

Definitions and some assumptions on which are based this investigation are given in the Section of preliminaries. The expressions of the covariance and spectral density functions of a nearly non stationary process are derived in the Section 3. Nearly nonstationary process having a pair of complex roots close to the unit circle is considered in detail in the Section 4 of this paper.

**2. Preliminaries.** Consider $\{X_t\}$ a process of discrete time $t \in Z$, described by a finite order ARMA$(p, q)$ model:

$$\alpha(B)X_t = \beta(B)\varepsilon_t, \quad t \in Z, \tag{1}$$

where $B$ stands for the backward shift operator defined as $BX_t = X_{t-1}$, and $\{\varepsilon_t, t \in Z\}$ is a sequence of independent identically distributed random variables with zero mean $E\varepsilon_t = 0$ and variance $E\varepsilon_t^2 = \sigma_\varepsilon^2$.

The polynomials are defined as

$$\alpha(z) = 1 - a_1 z - \ldots - a_p z^p, \tag{2}$$

$$\beta(z) = 1 + b_1 z + \ldots + b_q z^q, \tag{3}$$

such that they have no common zeroes and $\alpha(z) \neq 0$ for $|z| \leqslant 1$.

The class of ARMA models represents stationary series. A generalization of this class, which incorporates a wide range of nonstationary series is provided by the ARIMA$(p, d, q)$ processes, i.e., processes which, after differencing $d$ times, reduce to ARMA$(p, q)$ stationary processes. In other words, the polynomial $\alpha(z)$ of ARIMA$(p, d, q)$ process has $d$-fold roots equal to 1.

Let $R_X(\tau) = E\{X_t X_{t+\tau}\}$, $\tau = \pm 1, \pm 2, \ldots$ is the autocovariance function of the stationary sequence $\{X_t\}$ decaying sufficiently fast so that the spectral density function

$$h_X(w) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} Rx(\tau)e^{-iw\tau}, \quad -\pi \leqslant w \leqslant \pi, \tag{4}$$

exists. It is well known that the spectral densities $h_X(w)$ and $h_\varepsilon(w)$ of two sequences $\{X_t\}$ and $\{\varepsilon_t\}$ related by the equation (1) are expressed as

$$h_X(w) = \left|\frac{\beta(e^{-iw})}{\alpha(e^{-iw})}\right|^2 h_\varepsilon(w) = \left|\frac{\beta(e^{-iw})}{\alpha(e^{-iw})}\right|^2 \frac{\sigma_\varepsilon^2}{2\pi}, \quad -\pi \leqslant w \leqslant \pi, \quad (5)$$

assuming that $\alpha(e^{-iw})$ does not vanish in the range $(-\pi, \pi)$. Moreover, if any two stationary processes $\{X_t\}$ and $\{Y_t\}$ are related by the equation

$$X_t = \Phi(B)Y_t, \quad (6)$$

where $\Phi(z)$ is a transfer function of the linear filter represented by the polynomial of finite or infinite order, then their spectral density functions $h_X(w)$ and $h_Y(w)$ as well as their autocovariance functions $R_X(\tau)$ and $R_Y(\tau)$ are related by the equations

$$h_X(w) = |\Phi(e^{-iw})|^2 h_Y(w), \quad -\pi \leqslant w \leqslant \pi, \quad (7)$$

$$R_X(\tau) = \Phi(B)(\Phi(B^{-1})R_Y(\tau)). \quad (8)$$

Let $z_j = e^{-c_j + iw_j}$, $z_{j+1} = e^{-c_j - iw_j}$, $j = 1, 3, 5, \ldots, [p/2]$, are the pairs of the complex conjugate zeroes of the polynomial $z^p \alpha(z^{-1}) = z^p - a_1 z^{p-1} - \ldots - a_p$, otherwise called ARMA$(p, q)$ poles, satisfying the condition: $|z_j| < 1$ for all $j$. We are going to consider a situation when some $z_j$ are approaching to the unit circle. Notice that zeroes of $\alpha(z)$ polynomial are $z_j^{-1}$ located outside the unit circle.

ASSUMPTION 1. For $n = 1, 2, \ldots, N$, where $N$ is a large number, let

$$z_{1n} = e^{-c_1/n}, \qquad z_{2n} = -e^{-c_2/n},$$
$$z_{3n} = e^{-c_3/n + iw_0}, \qquad z_{4n} = e^{-c_3/n - iw_0}, \quad (9)$$

while the others $z_j$ are independent of $n$. Here $w_0$ is some constant, $0 \leqslant w_0 \leqslant \pi$.
For each $j = 1, 2, 3$ consider

$$\delta_{jn} = 1 - |z_{jn}|, \quad (10)$$

the value of which can be used to measure the effect of the closeness of $\alpha(z)$ zeroes to the unit circle on the spectral density and the covariance functions of process $\{X_t\}$.

ASSUMPTION 2. The polynomial $\alpha(z)$ has either $z_{1n}^{-1}$, either $z_{2n}^{-1}$ or a pair of $z_{3n}^{-1}, z_{4n}^{-1}$ as its zeroes, the other zeroes of $\alpha(z)$ being independent of $n$.

This leads to the polynomial $\alpha_n(z)$ instead of $\alpha(z)$ with parameters $a_{1n}, a_{2n}, \ldots, a_{pn}$ instead of $a_1, a_2, \ldots, a_p$. The $\beta(z)$ is not changed. Consequently we have to investigate the sequences of process $\{X_{tn}\}$ defined by

$$\alpha_n(B)X_{tn} = \beta(B)\varepsilon_t, \quad t \in Z, \ n = 1, 2, \ldots, N, \tag{11}$$

which are stationary for each fixed $n$ but as $n$ increases, the model (11) tends to ARIMA$(p-1, 1, Q)$ or ARIMA$(p-2, 2, q)$ models, representing non-stationary process. Many authors call such process nearly nonstationary or near-integrated process (including also the unit roots $z_j = 1$ and a mild explosive case $|z_j| \approx 1 + \varepsilon$).

Our aim is to analyse and explain the progressive deterioration of the covariance and spectral density functions of a stationary process in connection with the vanishing values $\delta_{1n}, \delta_{2n}, \delta_{3n}$ (as $n$ increases), each one having very specific effect on the second order characteristics of $\{X_t\}$. Therefore we have to investigate separately three cases:

$$
\begin{aligned}
a_n^{(1)}(B) &= (1 - z_{1n}B)\alpha^*(B), \\
a_n^{(2)}(B) &= (1 - z_{2n}B)\alpha^*(B), \\
a_n^{(3)}(B) &= (1 - z_{3n}B)(1 - z_{4n}B)\alpha^{**}(B),
\end{aligned}
\tag{12}
$$

where $\alpha^*(z)$ and $\alpha^{**}(z)$ are the polynomials of $(p-1)$ and $(p-2)$ order respectively with zeroes comfortably outside the unit circle. Now the expression (5) can be rewritten as

$$
\begin{aligned}
h_X(w; z_{jn}) &= \frac{1}{|1 - z_{jn}e^{-iw}|^2} \left| \frac{\beta(e^{-iw})}{\alpha^*(e^{-iw})} \right|^2 \frac{\sigma_\varepsilon^2}{2\pi} \\
&= \frac{1}{|1 - z_{jn}e^{-iw}|^2} h_Y(w), \quad -\pi \leqslant w \leqslant \pi, \ j = 1, 2,
\end{aligned}
\tag{13}
$$

where $h_Y(w)$ denotes the spectral density function of a well stationary ARMA$(p-1, q)$ process $\{Y_t\}$. The analogous expression for $h_X(w; z_{3n}, z_{4n})$ is easily derived with $\{Y_t\}$ being an ARMA$(p-2, q)$ process. Notice that for fixed $n$ and $|z_{jn}| < 1$ specified by (9) the function $|1 - z_{jn}e^{-iw}|^2 > 0$ can be treated as a spectral density function of the nearly non-stationary AR(1) process defined by

$$X_{0n} = 0,$$

$$X_{tn} - a_{jn}X_{t-1,n} = Y, \quad t = 1, 2, \ldots, n \tag{14}$$

$$a_{jn} = (-1)^{j+1}e^{-c_j/n}, \quad j = 1, 2,$$

where $n$ is number of observed values $x_1, x_2 \ldots, x_n$.

## 3. The covariance and spectral density functions of a nearly nonstationary process.

The behaviour of the covariance and spectral density functions of a nearly nonstationary process will be considered here, when some of ARMA$(p, q)$ poles $z_{jn}$ are tending to the unit circle.

Returning back to the equation (11) in the case $j = 1, 2$ we have

$$(1 - z_{jn}B)\alpha^*(B)X_{tn} = \beta(B)\varepsilon_t, \quad t \in Z \tag{15}$$

$$X_{tn} = \frac{1}{(1 - z_{jn}B)} \frac{\beta(B)}{\alpha^*(B)}\varepsilon_t = \frac{1}{(1 - z_{jn}B)}Y_t, \tag{16}$$

and for $j = 3$

$$X_{tn} = \frac{1}{1 - (z_{3n} + z_{4n})B + |z_{3n}|^2B^2}Y_t. \tag{17}$$

For any fixed $n$, $n \in (1, 2, \ldots, N)$, the processes $\{X_{tn}\}$ and $\{Y_t\}$ are both stationary having the well defined covariance functions $R_X(\tau)$ and $R_Y(\tau)$, what ensure the condition $|z_{jn}| < 1$. Having in mind the expressions (9), (10) we wish to express the functions $R_X(\tau; z_{jn})$, $h_X(w; z_{jn})$ by means of the corresponding functions of process $\{Y_t\}$ and $\delta_{jn}(j = 1, 2, 3)$ measuring the closeness to the unit circle.

**Lemma 1.** *Under assumptions 1 and 2 for fixed $n \in (1, 2, \ldots, N)$, the spectral density function $h_X(w; \delta_{jn})$ of process $\{X_{tn}\}$, satisfying (16) or (17) has the following expressions:*

$$h_X(w; \delta_{jn}) = \frac{h_Y(w)}{1 + 2(-1)^j(1 - \delta_{jn})\cos w + (1 - \delta_{jn})^2},$$
$$j = 1, 2 \quad -\pi \leqslant w \leqslant \pi, \tag{18}$$

$$h_X(w; \delta_{3n}) = \frac{h_Y(w)}{\prod\limits_{k=1}^{2}\left[1 - 2(1 - \delta_{3n})\cos\left[w + (-1)^k w_0\right] + (1 - \delta_{3n})^2\right]}$$
$$-\pi \leqslant w \leqslant \pi. \tag{19}$$

The proof of Lemma 1 follows immediately from (13) with (9) and (10). Notice that (18) is special case of (19) when $k = 1$ and $w_0 = -\pi$.

**Lemma 2.** *Under the assumptions* 1 *and* 2 *for fixed* $n \in (1, 2, \ldots, N)$ *the covariance function* $R_X(\tau; \delta_{jn})$ *of process* $\{X_{tn}\}$, *satisfying the equation* (16) *or* (17) *accordingly to* $j = 1, 2$ *or* $j = 3$, *is expressed in terms of the covariance function* $R_Y(\tau)$ *of a stationary ARMA process* $\{Y_t\}$ *by formulas:*

$$R_X(\tau; \delta_{jn}) = \sum_{k=-\infty}^{\infty} \frac{a_{jn}^{|k|}}{1 - a_{jn}^2} R_Y(\tau - k), \qquad (20)$$

*where* $a_{jn} = (-1)^{j+1}(1 - \delta_{jn})$, $j = 1, 2$,

$$R_X(\tau; \delta_{3n}) = \sum_{k=-\infty}^{\infty} \gamma_{kn} R_Y(\tau - k), \qquad (21)$$

*where*

$$\gamma_{kn} = \frac{\left(1 + e^{-2c_3/n}\right) e^{-c_3|k|n} \sin(kw_0 + \psi_n)}{\left(1 + e^{-2c_3/n}\right)\left(1 - 2e^{-2c_3/w}\cos 2w_0 + e^{-4c_3/n}\right)\sin \psi_n},$$

$$\mathrm{tg}\,\psi_n = \frac{1 + e^{-2c_3/n}}{1 - e^{-2c_3/n}}\mathrm{tg}w_0, \quad e^{-c_3/n} = 1 - \delta_{3n}.$$

*Proof.* We shall sketch the essential points to prove the validity of (20) for $R_X(w; \delta_{1n})$. In the case of $\delta_{2n}$ or $\delta_{3n}$ the proof is analogous. Notice that $a_{1n} > 0$ but $a_{2n} < 0$ and the same expression (20) of $R_X(\tau; \delta_{jn})$ describes different behaviour of the covariance function.

Applying the formula (8) to (16) we have

$$R_X(\tau; a_{1n}) = \frac{1}{(1 - a_{1n}B)(1 - a_{1n}B^{-1})} R_Y(\tau)$$

$$= \frac{1}{(1 - a_{1n}B^{-1})} \sum_{k=0}^{\infty} (a_{1n}B)^k R_Y(\tau)$$

$$= \frac{1}{(1 - a_{1n}B^{-1})} \sum_{k=0}^{\infty} (a_{1n})^k R_Y(\tau - k)$$

$$= \sum_{u=0}^{\infty} (a_{1n})^u B^{-u} \sum_{k=0}^{\infty} a_{1n}^k R_Y(\tau - k)$$

$$= \sum_{u=-\infty}^{0} a_{1n}^u B^{-u} \sum_{k=0}^{\infty} a_{1n}^k R_Y(\tau - k)$$

$$= \frac{1}{1 - a_{1n}^2} \sum_{k=-\infty}^{\infty} a_{1n}^{|k|} R_Y(\tau - k). \tag{22}$$

The same result could be derived from the Fourier transform properties and the expression (13). The right hand side of (22) is a convolution of two covariance functions: $R_Y(\tau)$ and covariance function of AR(1) process. Following the same arguments, the expression (21) is derived because $\gamma_{kn}$ actually is a covariance function of AR(2) process with complex conjugate characteristic roots $z_{3n}$ and $z_{4n}$ and AR coefficients $a_{1n} = -(z_{3n} + z_{4n})$, $a_{2n} = z_{3n} \cdot z_{4n}$.

The correlation function $\rho_{AR}(k)$ of such AR(2) process has the expression (Priestley, 1981)

$$\rho_{AR}(k) = a_{2n}^{k/2} \frac{\sin(kw_0 + \psi_n)}{\sin \psi_n}, \tag{23}$$

and its variance is expressed as

$$
\begin{aligned}
\sigma_{AR}^2(n) &= \frac{1 + a_{2n}}{(1 - a_{2n})[(1 + a_{2n})^2 - a_{1n}^2]} \\
&= \frac{1 + e^{-2c_3/n}}{(1 - e^{-2c_3/n})[1 - 2e^{-2c_3/n} \cos 2w_3 + e^{-4c_3/n}]}. \tag{24}
\end{aligned}
$$

Then from (18), (19) and (23), (24) in addition to the well known fact that a product of two spectral densities in time domain corresponds to the convolution of corresponding covariance functions, leads to (21).

**Theorem 1.** *Under assumptions 1 and 2, when n increases, $\delta_{jn} \to 0$, $j = 1, 2, 3$ and the spectral density functions $h_X(w; \delta_{jn})$ (18), (19) tend to the following functions*

$$h_X(w; \delta_{1n}) \to \frac{1}{4 \sin^2 \frac{w}{2}} h_Y(w), \quad -\pi \leqslant w \leqslant \pi; \tag{25}$$

$$h_X(w; \delta_{2n}) \to \frac{1}{4 \cos^2 \frac{w}{2}} h_Y(w), \quad -\pi \leqslant w \leqslant \pi; \tag{26}$$

$$h_X(w; \delta_{3n}) \to \frac{1}{4(\cos w - \cos w_0)^2} h_Y(w), \quad -\pi \leqslant w \leqslant \pi; \tag{27}$$

*while the process $\{X_{tn}\}$ itself as $\delta_{jn} \to 0$, $n \to \infty$ is described by difference equations:*

$$\alpha^*(B)\Delta_0 X_t = \beta(B)\varepsilon_t, \tag{28}$$

$$\alpha^*(B)\Delta_\pi X_t = \beta(B)\varepsilon_t, \tag{29}$$

$$\alpha^{**}(B)\Delta_{\pm w_0}X_t = \beta(B)\varepsilon_t, \tag{30}$$

*where we denote by $\Delta$ the limiting operator of $(1 - z_{jn}B)$ as $|z_{jn}| \to 1$. The subindex of $\Delta$ indicates the location of AR unit pole in complex plane with the corresponding frequency equal to $0$, or $\pi$, $\pm w_0$.*

*Proof.* The relationships (25)–(27) follow immediately from (8), (19) as $\delta_{jn} \to 0$. The coefficients $\gamma_{kn}$ in the expression (21) tend to $\cos w_0 k$ when $n$ increases and $\delta_{3n} \to 0$.

REMARK 1. The functions at the right hand side of (25), (26), (27) do not exist at the points $w = 0$, $w = \pm\pi$, $w = w_0$ correspondingly but are well defined for others $w \in [-\pi, \pi]$.

REMARK 2. The usual notation of the integrated ARMA process or ARIMA $(p, d, q)$ in the general case of $k$ zeros of orders $d_1, d_2, \ldots, d_k$ should be written as

$$\alpha(B)\Delta_{w_1,w_2,\ldots,w_k}^{d_1,d_2,\ldots,d_k}X_t = \beta(B)\varepsilon_t, \tag{31}$$

$$d_1 + d_2 + \ldots + d_k = d$$

## 4. Inference for nearly nonstationary ARMA processes.

In the previous section we have seen that $\delta_{jn}$ measuring the closeness of AR zeros to the unit circle, plays an important role in the behaviour of the second order characteristics of an ARMA process. However, in observed time series we are frequently uncertain whether the process has a root equal to the unit or it is close to the unit. The discriminatory power of tests for the presence of a unit root is rather low against such alternatives. We need to estimate $\delta_{jn}$ by calculating $\hat{\delta}_{jn}$ from observed values $x_1, x_2, \ldots, x_n$ of ARMA process, covering the possibility of $\delta_{jn} = 0$ and to develop an asymptotic theory for $g(n)(\hat{\delta}_{jn} - \delta_{jn})$ distribution. Luckily there is an abundance of similar results (Van der Meer *et al.*, 1993; Yap and Reinsel, 1995) for nearly nonstationary AR(1) processes with possibility to extend them for a more general class of processes.

### 4.1. Real root case.

Let us consider the case $\delta_{1n}$ when one real root of $\alpha_n(z) = 0$ tends to 1. Consider $\{X_{tn}, t \in Z\}$ for each $n \geqslant 1$, a process given by

$$X_{0n} = 0,$$

$$X_{tn} = a_{1n}X_{t-1,n} + Y_t, \quad t = 1, 2, \ldots, n, \tag{32}$$

$$a_{1n} = e^{-c_1/n}, \tag{33}$$

where $a_{1n}$ is an unknown parameter related to $\delta_{1n} = 1 - e^{-c_1/n}$, $\{Y_t\}$ is a stationary ARMA(p-1,q) process without roots close to the unit circle. The least-squares estimator of the parameter $\delta_{1n}$, based on observations $x_{1n}, x_{2n}, \ldots, x_{nn}$ is given by

$$\hat{\delta}_{1n} = 1 - \hat{a}_{1n} = 1 - \sum_{k=1}^{n} x_{kn} \cdot x_{k+1,n} \Big/ \sum_{k=1}^{n} x_{k-1,n}^2. \tag{34}$$

The value $\hat{a}_{1n}$ conveys vital information regarding the dependence structure of the process $\{X_t\}$; at the same time being an estimate of the partial correlation at lag1, the value $\hat{a}_{1n}$ extracts information about closeness to 1 of the "worst" characteristic root of AR polynomial.

Let $W(t)$, $t \in [0,1]$, denote the standard Wiener process and $\underset{\rightarrow}{D}$ denote the convergence in distribution, $\underset{\rightarrow}{P}$ – convergence in probability. It is known (Philips, 1987) that under quite weak conditions on $\{X_t\}$, including all Gaussian processes and many other finite order ARMA models with very general conditions on the underlying error term $\{\varepsilon_t\}$, it is true

$$n(\hat{a}_{1n} - a_{1n}) \overset{D}{\rightarrow} \int_0^1 Z_{c_1}(t)\, dW(t) + \frac{1}{2}(1 - \sigma_Y^2/\sigma^2)\Big/ \int_0^1 Z_{c_1}^2\, dt \overset{\triangle}{=} F(Z_{c_1}), \tag{35}$$

where $Z_{c_1}(t)$, $t \in [0,1]$ is an Ornstein-Uhlenbeck process defined as the solution of the stochastic differential equation

$$dZ_{c_1}(t) = c_1 Z_{c_1}(t)\, dt + dW(t), \quad Z_{c_1}(0) = 0. \tag{36}$$

When $c_1 = 0$ and $\{Y_t\}$ is independently and identically distributed with $EY_t = 0$, $EY_t^2 = \sigma$, the statement (35) reduce to known asymptotic result for AR(1) with a unit root (White, 1958; Fuller, 1976) because then $Z_{c_1}(t) = W(t)$ and $\sigma^2 = \sigma_Y^2$ in the formulae (35).

When $c_1 \neq 0$, Theorem 1 (Phillips, 1987) delivers the noncentral asymptotic theory for $\hat{a}_{1n}$, useful in studying the asymptotic power of tests for a unit root under the sequence of local alternatives given by $a = e^{c_1/n} \approx 1 + c_1/n$. When $c_1 < 0$ we have a local alternative that $\{X_t\}$ is stationary; when $c_1 > 0$, the local alternative is an explosive process $\{X_t\}$.

Suppose $a_{1n}$ is estimated by least-squares estimate given by (34) and the following associated statistics is calculated

$$\hat{s}_n^2 = \frac{1}{n} \sum_{t=2}^{n} (x_{tn} - \hat{a}_{1n}, x_{t,n-1})^2. \tag{37}$$

Then from Phillips (1987) we can reformulate the result for $\hat{\delta}_n$.

**Theorem 2.** *If* $\{X_{tn}, t \in Z\}$ *is a nearly nonstationary times series generated by (32), (33), then as* $n \to \infty$

$$(a) \quad n(\hat{\delta}_{jn} - \delta_{jn}) \xrightarrow{D} F(Z_{c_1}), \tag{38}$$

$$(b) \quad \hat{\delta}_{jn} \xrightarrow{P} 0, \quad \hat{s}_n^2 \xrightarrow{P} \sigma_Y^2 = 2\pi h_Y(0), \tag{39}$$

*where the functional* $F(Z_{c_1})$ *is given in (35).*

**4.2. Complex roots case.** It is proved in Ahtola and Tiao (1987b) that the estimates of $a_1$ and $a_2$ of more general model like (17) have exactly the same asymptotic distribution as the least squares estimates of AR(2). Therefore we shall formulate the results for a simple model

$$X_t = a_{1n}X_{t-1} + a_{2n}X_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2), \tag{40}$$

where $X_0 = X_{-1} = 0$; $a_{1n} = 2e^{-c_3/n}\cos w_0$, $a_{2n} = -e^{-2c_3/n}$, $t = 1, 2, \ldots, n$ and for simplicity here and further $c_3 \equiv c$.

First of all notice that from (40) we can write

$$X_t = \sum_{j=1}^{t} \psi_{j-1}\varepsilon_{t-j+1}, \tag{41}$$

where $\psi_0 = 1$, $\psi_1 = 2e^{-c/n}\cos w_0$ and for $j \geqslant 2$, $\psi_j = a_{1n}\psi_{j-1} + a_{2n}\psi_{j-2}$. Using expressions of $a_{1n}$ and $a_{2n}$ through complex conjugate roots $z_{3n}$, $z_{4n}$ after some elementary algebra we derive

$$\psi_k = \frac{e^{-ck/n}\sin(k+1)w_0}{\sin w_0}, \quad k = 0, 1, 2, \ldots \tag{42}$$

Denoting $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1})'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)'$ and $(n-1) \times n$ matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 0 \\ \psi_1 & 1 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \ldots & \cdot & \cdot & \cdot \\ \psi_{n-3} & \psi_{n-4} & \ldots & 1 & 0 & 0 \\ \psi_{n-2} & \psi_{n-3} & \ldots & \psi_1 & 1 & 0 \end{pmatrix} \tag{43}$$

we have

$$\mathbf{x} = \mathbf{T}\boldsymbol{\varepsilon}. \tag{44}$$

We shall first show that for model (40) it is true

$$n \begin{pmatrix} \hat{a}_{1n} - a_{1n} \\ \hat{a}_{2n} - a_{2n} \end{pmatrix} = \frac{1}{1 - d_n^2} \begin{pmatrix} 1 & -d_n \\ -d_n & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{1}{n} \sum_{t=1}^{n} x_t \varepsilon_t & \frac{1}{n^2} \sum_{t=1}^{n} x_{t-1}^2 \\ \frac{1}{n} \sum_{t=1}^{n} x_{t-2} \varepsilon_t & \frac{1}{n^2} \sum_{t=1}^{n} x_{t-1}^2 \end{pmatrix} + \begin{pmatrix} o_p(1) \\ o_p(1) \end{pmatrix}, (45)$$

where

$$d_n = \frac{a_{1n}}{1 - a_{2n}} = \frac{\cos w_0}{\operatorname{ch} c/n}.$$

*Proof.* The least squares estimators $(\hat{a}_{1n}, \hat{a}_{2n})$ are given by

$$\begin{pmatrix} \hat{a}_{1n} \\ \hat{a}_{2n} \end{pmatrix} = \mathbf{H}^{-1} \begin{pmatrix} \sum_{t=1}^{n} x_{t-1} x_t \\ \sum_{t=1}^{n} x_{t-2} x_t \end{pmatrix},$$

where $\qquad \mathbf{H} = \sum_{t=1}^{n} \begin{pmatrix} x_{t-1}^2 & x_{t-1} \cdot x_{t-2} \\ x_{t-1} \cdot x_{t-2} & x_{t-1} \end{pmatrix}.$ (46)

Noting that $x_t = a_{1n} x_{t-1} + a_{2n} x_{t-2} + \varepsilon_t$, we have from (42) that

$$n \begin{pmatrix} \hat{a}_{1n} - a_{1n} \\ \hat{a}_{2n} - a_{2n} \end{pmatrix} = \left( \sum_{t=1}^{n} x_{t-1}^2 \right) \mathbf{H}^{-1} \cdot \begin{pmatrix} \frac{1}{n} \sum_{t=1}^{n} x_{t-1} \varepsilon_t / \frac{1}{n^2} \sum_{t=1}^{n} x_{t-1}^2 \\ \frac{1}{n} \sum_{t=1}^{n} x_{t-2} \varepsilon_t / \frac{1}{n^2} \sum_{t=1}^{n} x_{t-1}^2 \end{pmatrix}. (47)$$

Now we show that

$$\mathbf{B} = \left( \sum_{t=1}^{n} x_{t-1}^2 \right) \begin{pmatrix} \sum_{t=1}^{n} x_{t-1}^2 & \sum_{t=1}^{n} x_{t-1} x_{t-2} \\ \sum_{t=1}^{n} x_{t-1} x_{t-2} & \sum_{t=1}^{n} x_{t-2}^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \frac{a_{1n}}{1 - a_{2n}} \\ \frac{a_{1n}}{1 - a_{2n}} & 1 \end{pmatrix} + \begin{pmatrix} 0 & o_p(1) \\ o_p(1) & o_p(1) \end{pmatrix}. (48)$$

Multiplying both sides of $x_t = a_{1n}x_{t-1} + a_{2n}x_{t-2} + \varepsilon_t$ by $x_{t-1}$ we have that

$$\sum x_t x_{t-1} = a_{1n} \sum x_{t-1}^2 + a_{2n} \sum x_{t-1}x_{t-2} + \sum \varepsilon_t x_{t-1},$$

$$\left( \sum_{t=1}^{n} x_{t-1}^2 \right)^{-1} \left[ \sum x_{t-1}(x_t - a_{2n}x_{t-2}) \right] = a_{1n}$$

$$+ \left( \sum_{t=1}^{n} x_{t-1}^2 \right)^{-1} \sum \varepsilon_t x_{t-1}.$$

So that

$$\frac{\sum x_{t-1}x_{t-2}}{\sum x_{t-1}^2} = \frac{a_{1n}}{1 - a_{2n}} + o_p(1), \tag{49}$$

and (48) is proved. Evidently, $\mathbf{B}^{-1} = \left( \sum_{t=1}^{n} x_{t-1}^2 \right) \mathbf{H}^{-1}$ and inserting this into (47) we have proved the result (45) or the same as:

$$n(\hat{a}_{1n} - a_{1n}) = \frac{1}{(1 - d_n^2)\dfrac{1}{n^2} \displaystyle\sum_{t=1}^{n} x_{t-1}^2} \left( \frac{1}{n} \sum_{t=1}^{n} x_{t-1}\varepsilon_t \right.$$

$$\left. - d_n \frac{1}{n} \sum_{t=1}^{n} x_{t-2}\varepsilon_t \right) + o_p(1), \tag{50}$$

$$n(\hat{a}_{2n} - a_{2n}) = \frac{1}{(1 - d_n^2)\dfrac{1}{n^2} \displaystyle\sum_{t=1}^{n} x_{t-1}^2} \left( -d_n \frac{1}{n} \sum_{t=1}^{n} x_{t-1}\varepsilon_t \right.$$

$$\left. + \frac{1}{n} \sum_{t=1}^{n} x_{t-2}\varepsilon_t \right) + o_p(1). \tag{51}$$

Quadratic forms from these expressions on the base of (42)–(44) can be represented as

$$\sum_{t=1}^{n} x_{t-1}\varepsilon_t = \varepsilon' A\varepsilon,$$

$$\sum_{t=1}^{n} x_{t-2}\varepsilon_t = \varepsilon' C\varepsilon, \tag{52}$$

$$\sum_{t=1}^{n} x_{t-1}^2 = \varepsilon T' \cdot T\varepsilon,$$

where the matrix $\mathbf{T}$ is given by (43) and the $n \times n$ matrices $\mathbf{A}$ and $\mathbf{C}$ are

$$
\mathbf{A} = \frac{1}{2}
\begin{pmatrix}
0 & 1 & \psi_1 & \cdot & \cdot & \cdot & \psi_{n-3} & \psi_{n-2} \\
1 & 0 & \psi_{n-2} & \cdot & \cdot & \cdot & \psi_{n-4} & \psi_{n-3} \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\psi_{n-4} & \psi_{n-5} & \psi_{n-6} & \cdot & \cdot & \cdot & 1 & \psi_1 \\
\psi_{n-3} & \psi_{n-4} & \psi_{n-5} & \cdot & \cdot & \cdot & 0 & 1 \\
\psi_{n-2} & \psi_{n-3} & \psi_{n-4} & \cdot & \cdot & \cdot & 1 & 0
\end{pmatrix},
$$

$$
\mathbf{C} = \frac{1}{2}
\begin{pmatrix}
0 & 0 & 1 & \psi_1 & \cdot & \cdot & \psi_{n-4} & \psi_{n-3} \\
0 & 0 & 0 & 1 & \cdot & \cdot & \psi_{n-5} & \psi_{n-4} \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\psi_{n-5} & \psi_{n-6} & \psi_{n-7} & \cdot & \cdot & 0 & 0 & 1 \\
\psi_{n-4} & \psi_{n-5} & \psi_{n-6} & \cdot & \cdot & 0 & 0 & 0 \\
\psi_{n-3} & \psi_{n-4} & \psi_{n-5} & \cdot & \cdot & 0 & 1 & 0
\end{pmatrix}. \qquad (53)
$$

Then (50) and (51) are rewritten as

$$
n(\hat{a}_{1n} - a_{1n}) = \frac{1}{(1 - d_n^2)} \frac{\frac{1}{n}\varepsilon'(\mathbf{A} - d_n\mathbf{C})\varepsilon}{\frac{1}{n^2}\varepsilon'\mathbf{T}'\mathbf{T}\varepsilon} + o_p(1), \qquad (54)
$$

$$
n(\hat{a}_{2n} - a_{2n}) = \frac{1}{(1 - d_n^2)} \frac{\frac{1}{n}\varepsilon'(\mathbf{C} - d_n\mathbf{A})\varepsilon}{\frac{1}{n^2}\varepsilon'\mathbf{T}'\mathbf{T}\varepsilon} + o_p(1). \qquad (55)
$$

In case $c = 0$, $d_n \equiv d = \cos\omega_0$, the matrices $\mathbf{A}$ and $\mathbf{C}$ are symmetric skew circulant matrices and their eigenvalues and eigenvectors are obtained explicitly (Ahtola and Tiao, 1987a) when complex roots are exactly on the unit circle the asymptotic distributions of statistics (54), (55) are derived. When $c \neq 0$, matrices $\mathbf{A}$ and $\mathbf{C}$ are no more circulant but all their diagonals are constant, each having a different damping factor $e^{-ck/n}$, $k = 1, 2, 3, \ldots, n$ and the limiting distribution of statistics (54), (55) unavailable.

Different approach (Van der Meer *et al.*, 1993) to handle nearly unstable models leads to parameter limit distributions:

$$
\begin{pmatrix} n(\hat{a}_{1n} - a_{1n}) \\ n(\hat{a}_{2n} - a_{2n}) \end{pmatrix} \xrightarrow{D} \frac{2}{S_Y^2} \begin{pmatrix} r_{YW}^+ \cos\omega_0 - r_{YW}^- \sin\omega_0 \\ -r_{YW}^+ \end{pmatrix} \qquad (56)
$$

where

$$
S_Y^2 = \int_0^1 (Y_1^2(t) + Y_2^2(t))\, dt,
$$

$$r^+_{YW} = \int_0^1 (Y_1(t)\,dW_1(t) + Y_2(t)\,dW_2(t)),$$

$$r^-_{YW} = \int_0^1 (Y_1(t)\,dW_2(t) - Y_2(t)\,dW_1(t)),$$

$W_1(t), W_2(t)$, $t \in [0,1]$ are independent real values standard Wiener process and the process $(Y_1(t), Y_2(t))$, $t \in [0,1]$ is the continuous time real valued AR(1) process given by

$$dY_1(t) = cY_1(t)\,dt + dW_1(t),$$
$$dY_2(t) = cY_2(t)\,dt + dW_2(t),$$

with initial values $Y_1(0) = Y_2(0) = 0$.

**4.3. Double roots equal to –1.** The case where the limit unstable model has double roots equal to –1, can be handled similarly, obtaining the limit distribution:

$$\begin{pmatrix} 0 & -n \\ -n^2 & n^2 \end{pmatrix} \begin{pmatrix} \hat{a}_{1n} - a_{1n} \\ \hat{a}_{2n} - a_{2n} \end{pmatrix} \xrightarrow{D} S^{-1} \begin{pmatrix} \int_0^1 \dot{Y}(t)\,dW(t) \\ \int_0^1 Y(t)\,dW(t) \end{pmatrix}, \tag{57}$$

where $(Y(t), \dot{Y}(t))$, $t \in [0,1]$ is the continuous time real valued AR(2) process given by

$$d\dot{Y}(t) = \left(2c_3\dot{Y}(t) + c_3^2 Y(t)\right) dt + dW(t), \tag{58}$$
$$dY(t) = \dot{Y}(t)\,dt, \tag{59}$$
$$Y(0) = \dot{Y}(0), \tag{60}$$

and

$$S = \begin{pmatrix} \int_0^1 (Y(t))^2 d(t) & \int_0^1 \dot{Y}(t)Y(t)d(t) \\ \int_0^1 Y(t)\dot{Y}(t)d(t) & \int_0^1 (\dot{Y}(t))^2 d(t) \end{pmatrix}. \tag{61}$$

**5. Comments.** We have seen that asymptotic distributions of parameters $n(\hat{a}_{1n} - a_{1n})$, $n(\hat{a}_{2n} - a_{2n})$ are heavily dependent on the parameters $c_1, c_2, c_3$ which also determine the values of $\delta_{jn}$, $j = 1,2,3$, introduced by (10) as a measure of closeness to the unstable model.

From the other side $\delta_{jn}$ are in one-to-one correspondence with the coefficients $a_{1n}, a_{2n}$, so the asymptotic distributions of statistics $(\hat{\delta}_{jn} - \delta_{jn})$, $j = 1,2,3$ can be easily derived.
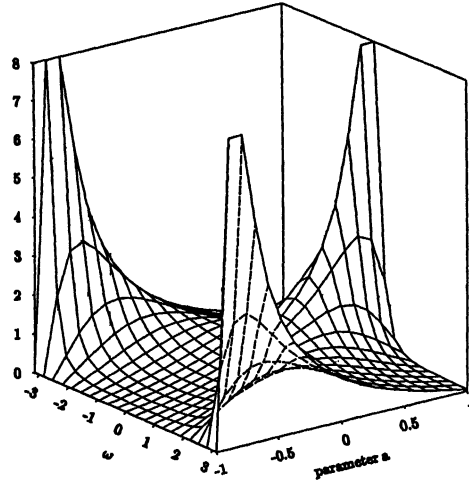
**Fig. 1.** The spectral density function of AR(1) as function of frequency $-\pi \leqslant w \leqslant \pi$ and parameter $-1 < a_{1n} < 1$.

The main idea to find a measure of closeness to the unstable model and to investigate their influence to the second order properties leads to investigation of contribution made by the parameters $c_1, c_2, c_3$, into the covariance function $R_X(\tau)$ and the spectral density function $h_X(w)$ of a nearly unstable process. Then we have the expressions.

$$h_X(w; c_j) = \frac{h_Y(w)}{1 + 2(-1)^j e^{-\frac{c_j}{n}} \cos w + e^{-\frac{2c_j}{n}}}, \quad j = 1, 2, \tag{62}$$

$$h_X(w; c_3) = \frac{h_Y(w)}{\prod\limits_{k=1}^{2} \left[ 1 - 2e^{-\frac{c_3}{n}} \cos[w + (-1)^k w_0] + 2e^{-\frac{2c_3}{n}} \right]}, \tag{63}$$

$$-\pi \leqslant w \leqslant \pi.$$

$$R_X(\tau; c_j) = \frac{1}{1 - e^{-\frac{2c_j}{n}}} \sum_{k=-\infty}^{\infty} \left[ (-1)^{j+1} e^{-\frac{c_j}{n}} \right]^{|k|} R_Y(\tau - k), \tag{64}$$

$$j = 1, 2,$$

and the contribution of the parameters $c_1, c_2, c_3$ is evident. The picture in Fig. 1. shows the dependence of the spectral density function $h_X(w; c_j)$ of AR(1) process on parameters $a_{1n}$, $-1 < a_{1n} < 1$, $a_{1n} = (-1)^j e^{-\frac{c_j}{n}}$, $j = 1, 2$. For the spectral density function $h_X(w, c_3)$ of AR(2) process we had to fix one of the

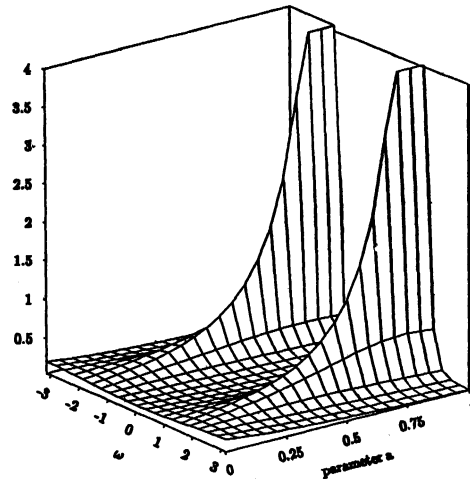**Fig. 2.** The spectral density function of AR(2), (comlex root case) as a function of $w$ and $a_{2n}$ $-\pi \leqslant w \leqslant \pi$, $0 < a_{2n} < 1$. The parameter $a_{1n}$ is fixed $a_{1n} = 0$.

parameters $a_{1n}, a_{2n}$ in order to have three dimension picture, Fig. 2. So the spectral density $h_X(w; c_3)$ has two clearly expressed peaks at the frequencies $-w_0$ and $w_0$. Concerning the covariance function $R_X(\tau; c_j)$, (64), its degeneration is mostly due to the factor $1 - e^{-\frac{2c_j}{n}}$ tending to zero as $n$ increases.

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# BEVEIK NESTACIONARŪS ARMA PROCESAI: ANTROSIOS EILĖS SAVYBĖS

## Nerutė KLIGIENĖ

Ištirti beveik nestacionariųjų ARMA procesų antrosios eilės momentai atvejais, kai AR lygties charakteringosios šaknys: (a) artėja prie 1; (b) artėja prie –1; (c) dvi kompleksinės jungtinės šaknys artėja prie vienetinio apskritimo. Tirtas kovariacinių ir spektrinio tankio funkcijų elgesys ir nustatytas iš esmės skirtingas jų išsigimimo pobūdis, sąlygojantis ARMA parametrų skirtingus pasiskirstymus (a), (b), (c) atvejais.