INFORMATICA, 1998, Vol. 9, No. 3, 365–386

A Mathematical Analysis of an Age-Sex-Space-Structured Population Dynamics Model with Random Mating and Females' Pregnancy

Vladas SKAKAUSKAS
Faculty of Mathematics, Vilnius University
Naugarduko 24, 2006 Vilnius, Lithuania
e-mail: vladas.skakauskas@maf.vu.lt

Received: May 1998

Abstract. We discuss an age-sex-structured population dynamics deterministic model taking into account random mating of sexes, females' pregnancy and its dispersal in whole space. This model can be derived from the previous one (Skakauskas, 1995) describing migration mechanism by the general linear elliptic operator of second order and includes the male, single (nonfertilized) female and fertilized female subclasses. Using the method of the fundamental solution for the uniformly parabolic second-order differential operator with bounded Hölder continuous coefficients we prove the existence and uniqueness theorem for the classic solution of the Cauchy problem for this model. In the case where dispersal moduli of fertilized females are not depending on age of the mated male we analyze population growth and decay.

Key words: population dynamics, random mating, gestation of males, pregnancy, migration.

1. Introduction

In the paper (Skakauskas, 1994) we have developed a general deterministic model for an age-sex-structured population dynamics taking into account random mating of sexes without formation of permanent male-female couples, female's pregnancy, possible destruction of the fetus (abortion), and female's sterility periods after abortion and delivery. The population is divided into five components: one male and four female, the latter four being the single (nonfertilized) female, fertilized female, female from sterility period after abortion, and female from sterility one following delivery. Each sex has three age-grades: pre-reproductive, reproductive, and post-reproductive. It is assumed that for each sex the commencement of each grade as well as the duration of the gestation and female's sterility periods are independent of individuals or time. Latter, in (Skakauskas, 1995), we generalized this model for the spatially dispersing population in whole space. Spatial dispersal mechanism in this model is described by an integral operator.

In the present paper we simplify the model in (Skakauskas, 1995) by neglecting abortion and female's sterility periods, replace the integral describing operator migration by the general linear elliptic differential one of second order and prove classical solvability of the initial problem for this model. The special case of the present model for the population dispersing in the whole space, where all vital rates of fertilized female are independent of age of mated partner, has been considered in (Skakauskas, 1997), and the existence and uniqueness theorems for the steady (i.e., time independent) and nonstationary cases have been proved.

This paper is organized as follows. In Section 3 we formulate the problem, Section 4 represents main hypotheses and results. In Section 5 we recall some classical results concerning unique solvability of the Cauchy problem for the linear differential parabolic operator of second order with parameter. Section 6 is devoted to proving the solvability theorem. In Sections 7 and 8 in the case where dispersal moduli of fertilized females are independent of age of mated males, we obtain upper estimates for population growth and extinction.

2. Notations

We follow the notations used in (Skakauskas, 1995):

τ_1, τ_2, τ_3: the ages of male, female, and embryo, respectively;

t: time;

E^m: Euclidean space (habitat of population) of dimension m;

$x = (x_1, x_2, \dots, x_m)$: the spatial position in E^m;

$u_1(x, t, \tau_1)$: the age-space density of males at age τ_1, location x and time t;

$u_2(x, t, \tau_2)$: the age-space density of single (nonfertilized) females at age τ_2, location x and time t;

$u_3(x, t, \tau_1, \tau_2, \tau_3)$: the age-space density of fertilized females at age τ_2, position x and time t whose embryo is at age τ_3 and that were fertilized by males at age τ_1;

$p(x, t, \tau_1, \tau_2)$: the density of probability to become fertilized for a female from the male-female pair formed of male at age τ_1 and female at age τ_2, at location x and time t;

$\nu_1(x, t, \tau_1)$ (resp. $\nu_2(x, t, \tau_2)$): the death rate of males at age τ_1 (resp. single females at age τ_2), position x and time t;

$\nu_3(x, t, \tau_1, \tau_2, \tau_3)$: the death rate of fertilized females at age τ_2, position x and time t whose embryo is at age τ_3 and that were fertilized by males at age τ_1; ;

$X(u_3)(x, t, \tau_2)$: the single female gain rate by the females which have had a delivery at age τ_2, position x and time t;

$Y(u_1)(x, t, \tau_2)u_2$: the single female loss due to conception at age τ_2, location x and time t;

$\sigma_1 = (\tau_{11}, \tau_{12}],\ 0 < \tau_{11} < \tau_{12} < \infty$: the female sexual activity interval, $\overline{\sigma}_1 = [\tau_{11}, \tau_{12}]$;

$\sigma_3 = (0, T],\ 0 < T < \infty$: the female gestation interval, $\overline{\sigma}_3 = [0, T]$;

$\sigma_2(\tau_3) = (\tau_{21} + \tau_3, \tau_{22} + \tau_3],\ 0 < \tau_{21} < \tau_{22} < \infty,\ \overline{\sigma}_2(\tau_3) = [\tau_{21} + \tau_3, \tau_{22} + \tau_3]$;

$\sigma_2(0), \sigma_2(T)$: the female fertilization and reproductivity intervals, respectively;

$n_1(x, t)$: the spatial density of males with ages from σ_1;

$b_1(x,t,\tau_1,\tau_2)$ and $b_2(x,t,\tau_1,\tau_2)$: the average numbers of male and female offspring, respectively, produced at time t at the position x by a fertilized female of characteristics (τ_1,τ_2,T);

$u_1^0(x,\tau_1)$, $u_2^0(x,\tau_2)$, $u_3^0(x,\tau_1,\tau_2,\tau_3)$: the initial distributions;

$\sigma=\sigma_1\times\sigma_2(T), \overline{\sigma}=\overline{\sigma}_1\times\overline{\sigma}_2(T),\ d\sigma=d\tau_1 d\tau_2$;

$\tau_2^0=0,\ \tau_2^1=\tau_{21},\ \tau_2^2=\min(\tau_{21}+T,\tau_{22}),\ \tau_2^3=\max(\tau_{21}+T,\tau_{22}),\ \tau_2^4=\tau_{22}+T,$ $\tau_2^5=\infty$;

$I=(0,\infty),\ \overline{I}=[0,\infty),\ I_4=(\tau_2^4,\infty),\ I_s=(\tau_2^s,\tau_2^{s+1}],\ s=\overline{0,3}$;

$I^*=(0,t^*],\ \overline{I}^*=[0,t^*],\ t^*<\infty$;

$Q_1=\big\{(x,t,\tau_1)\in E^m\times I\times I\big\},\ Q_2=\big\{(x,t,\tau_2)\in E^m\times I\times(I\setminus\bigcup_{s=1}^{4}\tau_2^s)\big\}$;

$Q_3=\big\{(x,t,\tau_1,\tau_2,\tau_3)\in E^m\times I\times\sigma_1\times\sigma_2(\tau_3)\times\sigma_3\big\}$;

$\overline{Q}_1=\overline{Q}_2=E^m\times\overline{I}\times\overline{I},\ \overline{Q}_3=E^m\times\overline{I}\times\overline{\sigma}_1\times\overline{\sigma}_2(\tau_3)\times\overline{\sigma}_3$;

$[u_2|_{\tau_2=\tau_2^j}]$: the jump of function u_2 at the plane $\tau_2=\tau_2^j$;

$\widehat{D}_1=\partial/\partial t+\partial/\partial\tau_1,\ \widehat{D}_2=\partial/\partial t+\partial/\partial\tau_2,\ \widehat{D}_3=\widehat{D}_2+\partial/\partial\tau_3$;

$D_1=\sqrt{2}\widetilde{D}_1,\ D_2=\sqrt{2}\widetilde{D}_2,\ D_3=\sqrt{3}\widetilde{D}_3$;

$\widetilde{D}_i,\ i=1,2,3$: the directional derivative in the positive direction of characteristics of the operator $\widehat{D}_i$;

$a_{ij}^k(x,t,\tau_k),\ a_i^k(x,t,\tau_k),\ k=1,2,\ a_{ij}^3(x,t,\tau_1,\tau_2,\tau_3),\ a_i^3(x,t,\tau_1,\tau_2,\tau_3),\ i,j=\overline{1,m}$: the space dispersal moduli of males ($k=1$), single females ($k=2$), and fertilized females;

$$L_k(x,t,\tau_k)=\sum_{i,j=1}^{m}a_{ij}^k\partial^2/\partial x_i\partial x_j+\sum_{i=1}^{m}a_i^k\partial/\partial x_i-\nu_k(x,t,\tau_k),\quad k=1,2,$$

$$L_3(x,t,\tau_1,\tau_2,\tau_3)=\sum_{i,j=1}^{m}a_{ij}^3\partial^2/\partial x_i\partial x_j+\sum_{i=1}^{m}a_i^3\partial/\partial x_i-\nu_3(x,t,\tau_1,\tau_2,\tau_3).$$

$C^0(E^m\times J_1\times\ldots\times J_s),\ J_s=(J_{s1},J_{s2}),J_{s1}<J_{s2}<\infty$: the Banach space of bounded continuous in $E^m\times J_1\times\ldots\times J_s$ functions $f(x,\xi_1,\xi_2,\ldots,\xi_s)$;

$C^{\alpha,0,\ldots,0}(E^m\times J_1\times\ldots\times J_s)$: the Banach space of functions $f(x,\xi_1,\xi_2,\ldots,\xi_s)$ belonging to $C^0(E^m\times J_1\times\ldots\times J_s)$, which are Hölder continuous in $(E^m\times J_1\times\ldots\times J_s)$ with exponent $\alpha\in(0,1)$ in x uniformly with respect to $(\xi_1,\xi_2,\ldots,\xi_s)$, i.e., having the finite Holder seminorm with respect to x (see [2]).

$C^{\alpha,\alpha/2,\ldots,\alpha/2}(E^m\times J_1\times\ldots\times J_s)$: the Banach space of functions $f(x,\xi_1,\xi_2,\ldots,\xi_s)$ belonging to $C^0(E^m\times J_1\times\ldots\times J_s)$, which are Hölder continuos in $(E^m\times J_1\times\ldots\times J_s)$ with exponent $\alpha\in(0,1)$ in x and $\alpha/2$ in $\xi_k,\ k=\overline{1,s}$, i.e., having the finite Hölder seminorm with respect to $x,\xi_1,\ldots,\xi_s$.

$C^{1,0,\ldots,0}(E^m\times J_1\times\ldots\times J_s)$: the Banach space of functions $f(x,\xi_1,\xi_2,\ldots,\xi_s)$ such that $\partial f/\partial x_i\in C^0(E^m\times J_1\times\ldots\times J_s),\ i=\overline{1,m}$.

For more details concerning population densities and vital rates we refer the reader to (Skakauskas, 1995).

3. Problem Formulation

The model to be discussed in this paper consists of the following nonlinear system of integrodifferential equations for u_1, u_2, u_3,

$$(D_1 - L_1)u_1 = 0 \quad \text{in } Q_1, \tag{1}$$

$$(D_2 - L_2)u_2 = X(u_3) - u_2 Y(u_1) \quad \text{in } Q_2, \tag{2}$$

$$Y(u_1) = \begin{cases} 0, & \tau_2 \notin \sigma_2(0), \\ n_1^{-1} \int_{\sigma_1} p u_1 \, d\tau_1, \ n_1 = \int_{\sigma_1} u_1 \, d\tau_1, & \tau_2 \in \sigma_2(0), \end{cases} \tag{3}$$

$$X(u_3) = \begin{cases} 0, & \tau_2 \notin \sigma_2(T), \\ \int_{\sigma_1} u_3|_{\tau_3 = T} \, d\tau_1, & \tau_2 \in \sigma_2(T), \end{cases} \tag{4}$$

$$(D_3 - L_3)u_3 = 0 \quad \text{in } Q_3, \tag{5}$$

which supplemented with the conditions

$$u_k|_{t=0} = u_k^0, \quad k = 1,2 \quad \text{in } E^m \times I,$$

$$u_3|_{t=0} = u_3^0 \quad \text{in } E^m \times \sigma_1 \times \sigma_2(\tau_3) \times \sigma_3, \tag{6}$$

$$u_k|_{\tau_k=0} = \int_\sigma b_k u_3|_{\tau_3=T} \, d\sigma, \quad k = 1,2 \quad \text{in } E^m \times I, \tag{7}$$

$$u_3|_{\tau_3=0} = p u_1 u_2 / n_1 \quad \text{in } E^m \times I \times \sigma_1 \times \sigma_2(0), \tag{8}$$

$$\left[u_2|_{\tau_2=\tau_2^s}\right] = 0, \quad s = \overline{1,4} \quad \text{in } E^m \times I, \tag{9}$$

describes evolution of the population with dispersal in whole space. In addition we assume that the initial distributions u_1^0, u_2^0, u_3^0 satisfy the following compatibility conditions

$$u_k^0|_{\tau_k=0} = \int_\sigma b_k|_{t=0} u_3^0|_{\tau_3=T} d\sigma, \quad k = 1,2 \quad \text{in } E^m,$$

$$u_3^0|_{\tau_3=0} = p|_{t=0} u_1^0 u_2^0 / \int_{\sigma_1} u_1^0 \, d\tau_1 \quad \text{in } E^m \times \sigma_1 \times \sigma_2(0). \tag{10}$$

As it follows from the foregoing, given functions ν_1, ν_2, ν_3, p, b_1, b_2, u_1^0, u_2^0, u_3^0 and the unknown ones u_1, u_2, u_3 must be positive-valued, otherwise they have no biological significance. Our purpose is to find u_1, u_2, u_3.

Observe that replacing $L_k\, u_k$, $k = 1, 2$ and $L_3\, u_3$ in (1)–(10) by

$$\int_{E^m} q_k(x, t, \tau_k, \xi) u_k(\xi, t, \tau_k)\, d\xi - u_k(x, t, \tau_k) \int_{E^m} q_k(\xi, t, \tau_k, x)\, d\xi, \quad k = 1, 2,$$

and

$$\int_{E^m} q_3(x, t, \tau_1, \tau_2, \tau_3, \xi) u_3(\xi, t, \tau_1, \tau_2, \tau_3) d\xi \\ - u_3(x, t, \tau_1, \tau_2, \tau_3) \int_{E^m} q_3(\xi, t, \tau_1, \tau_2, \tau_3, x) d\xi,$$

respectively, with given nonnegative q_1, q_2, q_3 we obtain a model analyzed in (Skakauskas, 1995).

4. Hypotheses and Main Results

Unless otherwise stated, the assumptions listed in this section hold throughout the paper:

(H_1) $p(x, t, \tau_1, \tau_2) \in C^{1,0,0,0}(E^m \times \overline{I} \times \overline{\sigma}_1 \times \overline{\sigma}_2(0))$ *is a nonnegative function with a compact support in* x *(*supp $p(\cdot, t, \tau_1, \tau_2)$*) and such that* $pt^{-1/2} \in C^{0,0,0,0}(E^m \times [0, \varepsilon] \times \overline{\sigma}_1 \times \overline{\sigma}_2(0))$ *for any small* $\varepsilon > 0$*;*

(H_2) $b_k(x, t, \tau_1, \tau_2) \in C^0(E^m \times \overline{I} \times \overline{\sigma}_1 \times \overline{\sigma}_2(T)),\ k = 1, 2$ *are nonnegative functions;*

(H_3) $\nu_k(x, t, \tau_k) \in C^{\alpha,0,0}(\overline{Q}_k)$*,* $k = 1, 2$*,* $\nu_3(x, t, \tau_1, \tau_2, \tau_3) \in C^{\alpha,0,0,0,0}(\overline{Q}_3)$ *are nonnegative functions;*

(H_4) $a^k_{ij}(x, t, \tau_k) \in C^{\alpha,\alpha/2,\alpha/2}(\overline{Q}_k)$*,* $a^k_i(x, t, \tau_k) \in C^{\alpha,0,0}(\overline{Q}_k)$*,* $k = 1, 2$*,* $i, j = \overline{1, m}$ *and* $a^3_{ij}(x, t, \tau_1, \tau_2, \tau_3) \in C^{\alpha,\alpha/2,\alpha/2,\alpha/2,\alpha/2}(\overline{Q}_3)$*,* $a^3_i(x, t, \tau_1\tau_2, \tau_3) \in C^{\alpha,0,0,0,0}(\overline{Q}_3),\ i = \overline{1, m}$ *are such that operators* $L_1,\ L_2,\ L_3$ *are uniformly elliptic* (see Garoni and Menaldi, 1992; Ladyzhenskaya *et al.*, 1967)*;*

(H_5) $u^0_1(x, \tau_1) \in C^0(E^m \times \overline{I})$ *is positive and* $u^0_2(x, \tau_2) \in C^0(E^m \times \overline{I})$*,* $u^0_3(x, \tau_1, \tau_2, \tau_3) \in C^0(E^m \times \overline{\sigma}_1 \times \overline{\sigma}_2(\tau_3) \times \overline{\sigma}_3)$ *are nonnegative functions verifying* (10)*.*

Now we list theorems for solvability of model (1)–(10), population growth and its decay which will be proved in Sections 6 and 7.

Theorem 1. *Under the hypotheses* (H_1)–(H_5) *problem* (1)–(10) *has for* $t \in \overline{I}^*$ *a unique nonnegative classic solution* (see Garoni and Menaldi, 1992; Ladyzhenskaya *et al.*, 1967) *such that* $u_1 \in C^0(E^m \times \overline{I}^* \times \overline{I})$*,* $u_2 \in C^0(E^m \times \overline{I}^* \times \overline{I})$*,* $u_3 \in C^0(E^m \times \overline{I}^* \times \overline{\sigma}_1 \times \overline{\sigma}_2(\tau_3) \times \overline{\sigma}_3)$*.*

Let us introduce the following notions

$$\widehat{b} = \max\Big\{ \int_{\sigma_2(T)} \sup_{E^m\times\overline{I}\times\overline{\sigma}_1} b_1\,d\tau_2,\ \int_{\sigma_2(T)} \sup_{E^m\times\overline{I}\times\overline{\sigma}_1} b_2\,d\tau_2 \Big\},$$

$$\widehat{p} = \sup_{E^m\times\overline{I}\times\overline{\sigma}_1\times\overline{\sigma}_2(0)} p,$$

$$\widehat{u} = \max\Big\{ \sup_{E^m\times\overline{I}} u_1^0,\ \sup_{E^m\times\overline{I}} u_2^0 \Big\}, \quad \widehat{u}_3 = \sup_{E^m\times\overline{\sigma}_2(T)} \int_{\sigma_1} u_3^0|_{\tau_3=T}\,d\tau_1.$$

$$\widetilde{\nu}_2 = \inf_{E^m\times\overline{I}\times(\overline{I}_2\cup\overline{I}_3)} \nu_2,$$

$$Q_*^3 = \{(x,t,\tau_1,\tau_2,\tau_3)\colon\ x\in E^m,\ 0<t\leqslant\tau_3,\ \tau_1\in\sigma_1,\ \tau_2\in\sigma_2(\tau_3),\ \tau_3\in\sigma_3\},$$

$$Q^{3*} = \{(x,t,\tau_1,\tau_2,\tau_3)\colon\ x\in E^m,\ t>\tau_3,\ \tau_1\in\sigma_1,\ \tau_2\in\sigma_2(\tau_3),\ \tau_3\in\sigma_3\}.$$

Theorem 2. *Let* (H$_1$)–(H$_5$) *hold, assume that* $a_{ij}^3,\ a_i^3,\ i,j=\overline{1,m}$ *are independent of* τ_1, *and let* $\widetilde{\nu}_2>0$.

Then

(i) $$\int_{\sigma_1} u_3\,d\tau_1 \leqslant \begin{cases} \widehat{u}_3 & \text{in } Q_{3*}, \\ \widehat{p}\sup\limits_{y\in E^m} u_2(y,t-\tau_3,\tau_2-\tau_3) & \text{in } Q_3^* \quad \text{for } t-\tau_3\in\overline{I}^*, \end{cases}$$

(ii) $u_k \leqslant \delta\gamma^s\widehat{u}$ *for* $t\in(sT,(s+1)T]\cap[0,t^*]$, $x\in E^m$, $\tau_k\in\overline{I}$,

with $s=0,1,\ldots,\ k=1,2$, *and* $\gamma=\max(\widehat{b}\widehat{p},1,\ \widehat{p}/\widetilde{\nu}_2)$, $\delta=\max\big(\widehat{b}\widehat{u}_3/\widehat{u},1,\widehat{u}_3/\widehat{u}\widetilde{\nu}_2\big)$ *(or more roughly* $u_k\leqslant\delta\widehat{u}\gamma^{t/T}$*).*

Define

$$q=\widehat{b}\widehat{u}_3/\widehat{u}, \quad \widetilde{\nu}=\min\big(\inf_{\overline{Q}_1}\nu_1,\quad \inf_{\overline{Q}_2}\nu_2\big),$$

$$\omega_0=\big\{(x,t,\xi)\colon\ x\in E^m,\quad 0\leqslant t\leqslant\xi,\ \xi\in\overline{I}\big\},$$

$$\omega_s=\big\{(x,t,\xi)\colon\ x\in E^m,\quad (s-1)\tau_k^4<t-\xi\leqslant s\tau_2^k,\ t\leqslant t^*,\ \xi\in\overline{I}\big\},$$
$$s=1,2\ldots.$$

Theorem 3. *Assume the hypotheses of Theorem 2 hold and let* $\widehat{b}\widehat{p}\leqslant q\leqslant\min(1,\widetilde{\nu}\widehat{b})$, $\widetilde{\nu}>0$. *Then:*

(i) $$\int_{\sigma_1} u_3\,d\tau_1 \leqslant \begin{cases} \widehat{u}_3 & \text{in } Q_{3*}, \\ \widehat{p}\sup_{y\in E^m} u_2(y,t-\tau_3,\tau_2-\tau_3) & \text{in } Q_3^* \quad \text{for } t-\tau_3\in\overline{I}^*, \end{cases}$$

(ii) $$\max\Big\{\sup_{\omega_s} u_1, \sup_{\omega_s} u_2\Big\} \leqslant \widehat{u}q^s.$$

COROLLARY 1. Let assumptions of Theorem 3 hold. If $q < 1$, then population vanishes as t increases.

5. Some Properties of Parabolic Operators of Second Order

In this section we collect some classical results concerning the solvability and uniqueness of the Cauchy problem for the linear differential parabolic operator of second order (see Garoni and Menaldi, 1992; Ladyzhenskaya *et al.*, 1967).

Lemma 1. *Let*

$$\Lambda(x,t,\beta) = \partial/\partial t - \sum_{i,j=1}^{m} b_{ij}\partial^2/\partial x_i \partial x_j - \sum_{i=1}^{m} \widetilde{b}_i \partial/\partial x_i + b_0$$

be a uniformly parabolic operator depending on a parameter $\beta \in J = [\beta_1, \beta_2]$, $\beta_1 < \beta_2 < \infty$ *with coefficients satisfying the following conditions*

$$b_{ij}(x,t,\beta) \in C^{\alpha,\alpha/2,0}(E^m \times \overline{I}^* \times J), \quad i,j = \overline{1,m},$$
$$\widetilde{b}_i(x,t,\beta) \in C^{\alpha,0,0}(E^m \times \overline{I}^* \times J), \quad i = \overline{0,m},$$

and assume that

$$0 < u^0(x,\beta) \in C^0(E^m \times J),$$
$$0 < f(x,t,\beta) \in C^{\alpha,0,0}(E^m \times \overline{I}^* \times J), \quad 0 < \alpha < 1.$$

Then problem

$$\Lambda u = f \quad \text{in } E^m \times \overline{I}^* \times J,$$
$$u(x,0,\beta) = u^0 \quad \text{in } E^m \times J \tag{11}$$

has a unique positive in $E^m \times \overline{I}^* \times J$ *classic solution* (see Garoni and Menaldi, 1992)

$$u(x,t,\beta) = \int_{E^m} \Gamma(x,t;y,0;\beta) u^0(y,\beta)\, dy + \int_0^t d\tau \int_{E^m} \Gamma(x,t;y,\tau;\beta) f(y,\tau,\beta) dy, \tag{12}$$

where $\Gamma(x,t;y,\tau;\beta)$ *is the fundamental solution of the operator* $\Lambda(x,t,\beta)$.

Lemma 2 (The comparison principle (see Friedman, 1968)). *Assume* b_0', b_0'', f', f'' *and* $u^{0\prime}$, $u^{0\prime\prime}$ *verify* $0 \leqslant b_0'' \leqslant b_0'$, $0 \leqslant f' \leqslant f''$ *and* $u^{0\prime} \leqslant u^{0\prime\prime}$ *in* $E^m \times \overline{I}^* \times \overline{J}$ *and*

$E^m \times \overline{J}$*, respectively. Then the corresponding solutions of problem* (11) *are such that* $0 \leqslant u' \leqslant u''$.

REMARK 1. Let u^0, $\Lambda(x,t,\beta)$ and its coefficients be as in Lemma 1. Then the conclusion of Lemma 1 holds true for f verifying

$$f(x,t,\beta) \in C^0(E^m \times \overline{I}^* \times \overline{J}), \quad |f(x,t,\beta) - f(y,t,\beta)| \leqslant \kappa t^{-\gamma}|x-y|^\alpha$$

with κ a constant and constant $\gamma \in (0,1)$.

The proof is the same as that of Lemma 1 in (Garoni and Menaldi, 1992) and based on the estimate

$$\Big| \int_{E^m} F(x,t;y,\tau;\beta) f(y,\tau,\beta)\, dy \Big| \leqslant \kappa_1 (t-\tau)^{-(2-\alpha)/2} \tau^{-\gamma}$$
$$\text{for } F = \partial^2\Gamma/\partial x_i \partial x_j, \quad \partial\Gamma/\partial t,$$

which can be established by using the following inequality given in (Garoni and Menaldi, 1992; Ladyzhenskaya *et al.*, 1967)

$$\Big| \int_{E^m} F(x-y,y,t,\tau,\beta)\, dy \Big| \leqslant \kappa_1 (t-\tau)^{-(2-\alpha)/2}$$
$$\text{for } F = \partial^2\Gamma_0/\partial x_i \partial x_j, \quad \partial\Gamma_0/\partial t,$$

where κ_1 is a constant and Γ_0 signifies the parametrix

$$\Gamma_0(x-y,y,\ t\ ,\tau,\beta) = \Big\{ [4\pi(t-\tau)]^{m/2} \big(\det B(y,\tau)\big)^{1/2} \Big\}^{-1}$$
$$\times \exp\Big\{ -\{4(t-\tau)\}^{-1} \sum_{i,j=1}^{m} B^{ij}(y,\tau)\,(x_i-y_i)(x_j-y_j) \Big\},$$

B means the matrix with elements $b_{ij}(x,t,\beta)$, B^{-1} its inverse with elements B^{ij}.

We can also prove the estimate

$$\Big| \int_{E^m} \{\Gamma(x,t;y,\tau;\beta) - \Gamma(x',t;y,\tau;\beta)\}\, f(y,\xi,\beta)\, dy \Big|$$
$$\leqslant \kappa_1 |x-x'| (t-\tau)^{-1/2} \tag{13}$$

for $f \in C^0(E^m \times \overline{I}^* \times J)$, where κ_1 is a constant.

REMARK 2. Under the hypotheses of Remark 1, $u \in C^0(E^m \times \overline{I}^* \times J)$.

The proof is based on the estimate

$$|\Gamma(x,t;y,\tau;\beta'') - \Gamma(x,t;y,\tau;\beta')| < \varepsilon c(t-\tau)^{-m/2} \exp\left\{-C|x-y|^2/(t-\tau)\right\} \tag{14}$$

for every $\varepsilon > 0$ and sufficiently small $|\beta'' - \beta'|$, where c and C are two positive constants. This estimate can be proved by parametrix method, used in (Ladyzhenskaya *et al.*, 1967) for construction of $\Gamma(x,t;y,\tau;\beta)$, as follows. According to Ladyzhenskaya *et al.* (1967) we have

$$\Gamma(x,t;y,\tau;\beta) = \Gamma_0(x-y,y,t,\tau,\beta) + \int_\tau^t d\lambda \int_{E^m} \Gamma_0(x-\xi,\xi,t,\lambda,\beta) R(\xi,\lambda;y,\tau;\beta)\, d\xi,$$

$$R(x,t;\xi,\tau;\beta) + \int_\tau^t d\lambda \int_{E^m} K(x,t;y,\lambda;\beta) R(y,\lambda;\xi,\tau;\beta)\, dy + K(x,t;\xi,\tau;\beta) = 0,$$

$$K(x,t;y,\lambda;\beta) = \sum_{i,j=1}^m \big(b_{ij}(y,\lambda,\beta) - b_{ij}(x,t,\beta)\big) \frac{\partial^2}{\partial x_i \partial x_j} \Gamma_0(x-y,y,t,\lambda,\beta) + \Big\{ -\sum_{i=1}^m b_i(x,t,\beta)\partial/\partial x_i + b_0(x,t,\beta)\Big\} \Gamma_0(x-y,y,t,\lambda,\beta),$$

$$|K(x,t;y,\tau;\beta)| \leqslant c(t-\tau)^{-(m+2-\alpha)/2} \exp\left\{-C|x-y|^2/(t-\tau)\right\},$$

$$|R(x,t;\xi,\tau;\beta)| \leqslant c(t-\tau)^{-(m+2-\alpha)/2} \exp\left\{-C|x-\xi|^2/(t-\tau)\right\},$$

$$|\Gamma(x,t;\xi,\tau;\beta)| \leqslant c(t-\tau)^{-m/2} \exp\left\{-C|x-\xi|^2/(t-\tau)\right\}.$$

Letting

$$g(x,\xi,t,\tau,\beta'',\beta') = R(x,t;\xi,\tau;\beta'') - R(x,t;\xi,\tau;\beta'),$$

we obtain

$$g(x,\xi,t,\tau,\beta'',\beta') + \int_\tau^t d\lambda \int_{E^m} K(x,t;y,\lambda;\beta'') g(y,\xi,\lambda,\tau,\beta'',\beta') dy + \int_\tau^t d\lambda \int_{E^m} \big(K(x,t;y,\lambda;\beta'') - K(x,t;y,\lambda;\beta')\big) R(y,\lambda;\xi,\tau;\beta') dy + K(x,t;\xi,\tau;\beta'') - K(x,t;\xi,\tau;\beta') = 0,$$

which by

$$|\Gamma_0(x-y,y,t,\tau,\beta'') - \Gamma_0(x-y,y,t,\tau,\beta'|$$
$$\leqslant \varepsilon_1 c(t-\tau)^{-m/2}\exp\left\{-C|x-y|^2/(t-\tau)\right\}$$

and

$$|K(x,t;y,\lambda;\beta'') - K(x,t;y,\lambda;\beta')|$$
$$\leqslant \varepsilon_1 (t-\lambda)^{-(m+2-\alpha)/2}\exp\left\{-C|x-y|^2/(t-\lambda)\right\}$$

yields

$$|g(x,\xi,t,\tau,\beta'',\beta')| \leqslant \varepsilon_1 c(t-\tau)^{-(m+2-\alpha)/2}\exp\left\{-C|x-\xi|^2/(t-\tau)\right\}$$

for every $\varepsilon_1 > 0$ and sufficiently small $|\beta'' - \beta'|$. Combining these estimates proves (14) which, by (12) and due to continuity of u for fixed value of parameter, enables us to estimate difference $u(x'',t'',\beta'') - u(x',t',\beta'') + u(x',t',\beta'') - u(x',t',\beta')$. This ends the proof of Remark 2.

6. Proof of Theorem 1

Now we are in position to prove Theorem 1. We limit ourselves to the case of multiple deliveries including overlapping between successive generations, i.e., $T < \tau_{22} - \tau_{21}$, $\tau_2^2 = \tau_{21} + T, \tau_3^2 = \tau_{22}$. The opposite case can be considered by similar argument.

Set

$$Q_1 = Q_*^1 \cup Q^{1*},$$
$$E^m \times I \times I = \bigcup_{s=0}^{4} Q_k^2, \quad Q_k^2 = E^m \times I \times I_k = Q_{k*}^2 \cup Q_k^{2*}, \quad k = \overline{0,4},$$
$$Q_3 = Q_*^3 \cup Q^{3*},$$

where

$$Q_*^1 = \{(x,t,\tau_1)\colon\, x \in E^m,\ 0 < t \leqslant \tau_1,\ \tau_1 \in I\},$$
$$Q^{1*} = \{(x,t,\tau_1)\colon\, x \in E^m,\ t > \tau_1,\ \tau_1 \in I\},$$
$$Q_{k*}^2 = \left\{(x,t,\tau_2)\colon\, x \in E^m,\ 0 < t \leqslant \tau_2 - \tau_2^k,\ \tau_2 \in I_k\right\},$$
$$Q_k^{2*} = \left\{(x,t,\tau_2)\colon\, x \in E^m,\ t > \tau_2 - \tau_2^k,\ \tau_2 \in I_k\right\},$$
$$Q_*^3 = \{(x,t,\tau_1,\tau_2,\tau_3)\colon\, x \in E^m,\ 0 < t \leqslant \tau_3,\ \tau_1 \in \sigma_1,\ \tau_2 \in \sigma_2(\tau_3),\ \tau_3 \in \sigma_3\},$$
$$Q^{3*} = \{(x,t,\tau_1,\tau_2,\tau_3)\colon\, x \in E^m,\ t > \tau_3,\ \tau_1 \in \sigma_1,\ \tau_2 \in \sigma_2(\tau_3),\ \tau_3 \in \sigma_3\}.$$

Let $\tau_1 = t + \eta_1$ and $\tau_2 = t + \eta_2$ be characteristics of the operators $\widehat{D}_1$ and $\widehat{D}_2$, respectively, and assume that $\tau_2 = t + \eta_3$, $\tau_3 = t + \eta_4$ mean the characteristics of $\widehat{D}_3$. Here $\eta_1, \eta_2, \eta_3, \eta_4$ denote parameters of the characteristics. Letting

$$
\begin{aligned}
&L_1(x,t,t+\eta_1) = L_{1*}(x,t,\eta_1), \quad u_1(x,t,t+\eta_1) = u_{1*}(x,t,\eta_1) \quad \text{in } Q^1_*,\\
&L_1(x,\tau_1-\eta_1,\tau_1) = L^*_1(x,\tau_1,-\eta_1),\\
&u_1(x,\tau_1-\eta_1,\tau_1) = u^*_1(x,\tau_1,-\eta_1) \quad \text{in } Q^{1*},\\
&L_2(x,t,t+\eta_2) = L_{2*}(x,t,\eta_2), \quad u_2(x,t,t+\eta_2) = u_{2*}(x,t,\eta_2),\\
&Y(x,t,t+\eta_2) = Y_*(x,t,\eta_2), \quad X(x,t,t+\eta_2) = X_*(x,t,\eta_2) \quad \text{in } \bigcup_{k=0}^{4} Q^2_{k*},\\
&L_2(x,\tau_2-\eta_2,\tau_2) = L^*_2(x,\tau_2,-\eta_2), \quad u_2(x,\tau_2-\eta_2,\tau_2) = u^*_2(x,\tau_2,-\eta_2),\\
&Y(x,\tau_2-\eta_2,\tau_2) = Y^*(x,\tau_2,-\eta_2),\\
&X(x,\tau_2-\eta_2,\tau_2) = X^*(x,\tau_2,-\eta_2) \quad \text{in } \bigcup_{k=0}^{4} Q^{2*}_k,\\
&L_3(x,t,\tau_1,t+\eta_3,t+\eta_4) = L_{3*}(x,t,\tau_1,\eta_3,\eta_4),\\
&u_3(x,t,\tau_1,t+\eta_3,t+\eta_4) = u_{3*}(x,t,\tau_1,\eta_3,\eta_4) \quad \text{in } Q^3_*,\\
&L_3(x,\tau_3-\eta_4,\tau_1,\tau_3+\eta_3-\eta_4,\tau_3) = L^*_3(x,\tau_3,\tau_1,-\eta_4,\eta_3-\eta_4),\\
&\quad u_3(x,\tau_3-\eta_4,\tau_1,\tau_3+\eta_3-\eta_4,\tau_3) = u^*_3(x,\tau_3,\tau_1,-\eta_4,\eta_3-\eta_4) \quad \text{in } Q^{3*},\\
&\widetilde{L}_{2*} = L_{2*} - Y_*, \quad \widetilde{L}^*_2 = L^*_2 - Y^*,
\end{aligned}
$$

and taking (1)–(10) on the respective characteristics we obtain:

$$
\begin{aligned}
&(\partial/\partial t - L_{1*})u_{1*} = 0 \text{ in } \quad Q^1_*, \quad u_{1*}(x,0,\eta_1) = u^0_1(x,\eta_1),\\
&(\partial/\partial \tau_1 - L^*_1)u^*_1 = 0 \quad \text{in } Q^{1*}, \quad u^*_1(x,0,-\eta_1) = u_1(x,-\eta_1,0),\\
&(\partial/\partial t - \widetilde{L}_{2*})u_{2*} = X_* \quad \text{in } \bigcup_{k=0}^{4} Q^2_{k*}, \quad u_{2*}(x,0,\eta_2) = u^0_2(x,\eta_2),\\
&(\partial/\partial \tau_2 - \widetilde{L}^*_2)u^*_2 = X^* \text{ in } Q^{2*}_k,\\
&u^*_2(x,\tau^k_2,-\eta_2) = u_2(x,\tau^k_2-\eta_2,\tau^k_2), \quad k=\overline{0,4},\\
&(\partial/\partial t - L_{3*})u_{3*} = 0 \quad \text{in } Q^3_*,\\
&u_{3*}(x,0,\tau_1,\eta_3,\eta_4) = u^0_3(x,\tau_1,\eta_3,\eta_4),\\
&(\partial/\partial \tau_3 - L^*_3)u^*_3 = 0 \quad \text{in } Q^{3*},\\
&u^*_3(x,0,\tau_1,-\eta_4,\eta_3-\eta_4) = u_3(x,-\eta_4,\tau_1,\eta_3-\eta_4,0).
\end{aligned}
\tag{15}
$$

By virtue of (H_3)–(H_5), operators L_{k*}, L^*_k, $k = 1,2,3$ and initial distributions $u_{1*}(x,0,\eta_1)$, $u_{2*}(x,0,\eta_2)$, $u_{3*}(x,0,\tau_1,\eta_3,\eta_4)$ satisfy all the conditions of Remark 1. If Y_*, Y^*, $u_1(x,-\eta_1,0)$, $u_2(x,\tau^k_2-\eta_2,\tau^k_2)$, $u_3(x,-\eta_4,\tau_1,\eta_3-\eta_4,0)$, and X_*, X^* are known and satisfy all the conditions of Remark 1, then system (15) degenerates into separate problems for u_{1*}, u^*_1, u_{2*}, u^*_2, u_{3*}, u^*_3, respectively, of type (11).

Denoting by

$$
\Gamma^1_*(x,t;y,\xi;\eta_1), \quad \Gamma^{1*}(x,\tau_1;y,\xi;-\eta_1),
$$

$$\Gamma^2_{k*}(x,t;y,\xi;\eta_2), \quad \Gamma^{2*}_k(x,\tau_2;y,\xi;-\eta_2),$$
$$\Gamma^3_*(x,t;y,\xi;\tau_1,\eta_3,\eta_4), \quad \Gamma^{3*}(x,\tau_3;y,\xi;\tau_1,-\eta_4,\eta_3-\eta_4)$$

the fundamental solutions of operators

$$\partial/\partial t - L_{1*}, \quad \partial/\partial\tau_1 - L_1^*, \quad \partial/\partial t - \widetilde{L}_{2*}, \quad \partial/\partial\tau_2 - \widetilde{L}_2^*,$$
$$\partial/\partial t - L_{3*}, \quad \partial/\partial\tau_3 - L_3^*$$

in Q^1_*, Q^{1*}, Q^2_{k*}, Q^{2*}_k, Q^3_*, Q^{3*}, respectively, from (15), applying general formula (12) to u_{k*}, u_k^*, $k=1,2,3$, we obtain the system

$$u_{1*}(x,t,\eta_1) = \int_{E^m} \Gamma^1_*(x,t;y,0;\eta_1)u_1^0(y,\eta_1)dy \quad \text{in } Q^1_*,$$

$$u_1^*(x,\tau_1,-\eta_1) = \int_{E^m} \Gamma^{1*}(x,\tau_1;y,0;-\eta_1)u_1(y,-\eta_1,0)dy \quad \text{in } Q^{1*},$$

$$u_{2*}(x,t,\eta_2) = \int_{E^m} \Gamma^2_{k*}(x,t;y,0;\eta_2)u_2^0(y,\eta_2)dy + \int_0^t d\xi \int_{E^m} \Gamma^2_{k*}(x,t;y,\xi;\eta_2)X_*(y,\xi,\eta_2)dy \quad \text{in } Q^2_{k*},$$

$$u_2^*(x,\tau_2,-\eta_2) = \int_{E^m} \Gamma^{2*}_k(x,\tau_2;y,\tau_2^k;-\eta_2)u_2(y,\tau_2^k-\eta_2,\tau_2^k)dy + \int_{\tau_2^k}^{\tau_2} d\xi \int_{E^m} \Gamma^{2*}_k(x,\tau_2;y,\xi;-\eta_2)X^*(y,\xi,-\eta_2)dy \quad \text{in } Q^{2*}_k,$$

$$u_{3*}(x,t,\tau_1,\eta_3,\eta_4) = \int_{E^m} \Gamma^3_*(x,t;y,0;\tau_1,\eta_3,\eta_4)u_3^0(y,\tau_1,\eta_3,\eta_4)dy \quad \text{in } Q^3_*,$$

$$u_3^*(x,\tau_3,\tau_1,-\eta_4,\eta_3-\eta_4) = \int_{E^m} \Gamma^{3*}(x,\tau_3;y,0;\tau_1,-\eta_4,\eta_3-\eta_4)u_3(y,-\eta_4,\tau_1,\eta_3-\eta_4,0)dy \quad \text{in } Q^{3*},$$

which by (4), (6)–(8) can be written as follows

$$u_1(x,t,\tau_1) = \int_{E^m} \Gamma^1_*(x,t;y,0;\tau_1-t)u_1^0(y,\tau_1-t)\,dy \quad \text{in } Q^1_*, \tag{16}$$

$$u_1(x,t,\tau_1) = \int_{E^m} \Gamma^{1*}(x,\tau_1;y,0;t-\tau_1)u_1(y,t-\tau_1,0)\,dy \quad \text{in } Q^{1*}, \tag{17}$$

$$u_2(x,t,\tau_2) = \int_{E^m} \Gamma^2_{k*}(x,t;y,0;\tau_2-t)u_2^0(y,\tau_2-t)\,dy \quad \text{in } Q^2_{k*}, \quad k=0,1,4, \tag{18}$$

$$u_2(x,t,\tau_2) = \int_{E^m} \Gamma^2_{k*}(x,t;y,0;\tau_2 - t)u^0_2(y,\tau_2 - t)\,dy + \int_0^t d\xi \int_{E^m} dy\Gamma^2_{k*}(x,t;y,\xi;\tau_2 - t)\int_{\sigma_1} u_3(y,\xi,\tau_1,\xi+\tau_2-t,T)\,d\tau_1 \quad \text{in } Q^2_{k*}, \quad k = 2,3, \tag{19}$$

$$u_2(x,t,\tau_2) = \int_{E^m} \Gamma^{2*}_k(x,\tau_2;y,\tau^k_2;t-\tau_2)u_2(y,\tau^k_2 + t - \tau_2,\tau^k_2)\,dy \quad \text{in } Q^{2*}_k, \quad k = 0,1,4, \tag{20}$$

$$u_2(x,t,\tau_2) = \int_{E^m} \Gamma^{2*}_k(x,\tau_2;y,\tau^k_2;t-\tau_2)u_2(y,\tau^k_2 + t - \tau_2,\tau^k_2)\,dy + \int_{\tau^k_2}^{\tau_2} d\xi \int_{E^m} dy\Gamma^{2*}_k(x,\tau_2;y,\xi;t-\tau_2) \times \int_{\sigma_1} u_3(y,\xi+t-\tau_2,\tau_1,\xi,T)\,d\tau_1 \quad \text{in } Q^{2*}_k, \quad k = 2,3, \tag{21}$$

$$u_3(x,t,\tau_1,\tau_2,\tau_3) = \int_{E^m} \Gamma^3_*(x,t;y,0;\tau_1,\tau_2 - t,\tau_3 - t) \times u^0_3(y,\tau_1,\tau_2 - t,\tau_3 - t)\,dy \quad \text{in } Q^3_*, \tag{22}$$

$$u_3(x,t,\tau_1,\tau_2,\tau_3) = \int_{E^m} \Gamma^{3*}(x,\tau_3;y,0;\tau_1,t-\tau_3,\tau_2-\tau_3) \times u_3(y,t-\tau_3,\tau_1,\tau_2-\tau_3,0)\,dy \quad \text{in } Q^{3*}, \tag{23}$$

$$u_3(x,t,\tau_1,\tau_2,0) = p(x,t,\tau_1,\tau_2)u_1(x,t,\tau_1)u_2(x,t,\tau_2)/n_1(x,t), \quad n_1 = \int_{\sigma_1} u_1\,d\tau_1, \tag{24}$$

$$u_1(x,t,0) = \int_\sigma b_1 u_3(x,t,\tau_1,\tau_2,T)\,d\sigma, \tag{25}$$

$$u_2(x,t,0) = \int_\sigma b_2 u_3(x,t,\tau_1,\tau_2,T)\,d\sigma. \tag{26}$$

We must add to (20) and (21) the continuity condition $[u|_{\tau_2=\tau_2^k}] = 0,\ k = \overline{1.4}$.

Now we will prove that (16)–(26) represent the solution of (1)–(10). Consider system (16)–(26) going along the axis t by step T. Since L_{1*}, L_{2*} in $Q^2_{0*} \cup Q^2_{4*}$ and L_{3*} satisfy the conditions of Remark 1, functions (16), (18) for $k = 0$ and (22) express positive u_1, u_2 and u_3 in Q^1_*, $Q^2_{0*} \cup Q^2_{4*}$ and Q^3_*, respectively. Hence, by virtue of (H$_1$) we observe that $n_1|_{\operatorname{suppa} p(\cdot,t,\tau_1,\tau_2)} \geqslant \widetilde{n}_1$ and

$$p(x,t,\tau_1,\tau_2)u_1(x,t,\tau_1)/n_1(x,t) \in C^0\left(E^m \times [0,\tau_{11}] \times \overline{\sigma}_1 \times \overline{\sigma}_2(0)\right),$$

where $\widetilde{n}_1$ is a positive constant, while from (22) by (13) it follows that

$$|u_3(x,t,\tau_1,\tau_2,\tau_3) - u_3(y,t,\tau_1,\tau_2,\tau_3)| \leqslant \kappa_1 |x-y| t^{-1/2} \quad \text{in } Q^3_*, \tag{27}$$

with κ_1 a constant.

Let $t \in [0,T]$ and assume $\omega_1 = E^m \times [0,T] \times \overline{I}$. By means of (25), (26), (H_2) and due to the continuity of u_3 (see Remark 1) we obtain continuos $u_1(x,t,0)$ and $u_2(x,t,0)$ $\forall (x,t) \in E^m \times [0,T]$. Now from (17) and (20) for $k=0$ we get continuos u_1 and u_2 in $Q^{1*} \cap \omega_1$ and $Q^{2*}_0 \cap \omega_1$, respectively. Then by (H_1) it follows that

$$\begin{aligned}
&Y(x,t,\tau_2) \in C^0\left(E^m \times [0,T] \times \overline{\sigma}_2(0)\right),\\
&|Y(x,t,\tau_2) - Y(y,t,\tau_2)| = \left| \int_0^1 (x-y)\cdot \nabla_z Y(z,t,\tau_2)|_{z=y+\gamma(x-y)}\, d\gamma \right|\\
&\qquad \leqslant |x-y| \sup_{E^m \times (0,T] \times \sigma_2(0)} |\nabla_x Y(x,t,\tau_2)|,
\end{aligned}$$

where ∇_x and $(x-y)\cdot\nabla_z$ mean the gradient operator and scalar product, respectively. Since

$$|\nabla_x Y(x,t,\tau_2)| \leqslant n_1^{-1} \left| \int_{\sigma_1} \left(u_1 \nabla_x p + p \nabla_x u_1\right) d\tau_1 \right| + n_1^{-2} |\nabla_x n_1| \int_{\sigma_1} p u_1\, d\tau_1,$$

we have:

$$n_1^{-1} \left| \int_{\sigma_1} u_1 \nabla_x p\, d\tau_1 \right| \leqslant \kappa_2$$

by (H_1) and because

$$n_1|_{\operatorname{suppa} p(\cdot,t,\tau_1,\tau_2)} \geqslant \widetilde{n}_2,$$

and

$$n_1^{-1} \int_{\sigma_1} p |\nabla_x u_1|\, d\tau_1 + n_1^{-2} |\nabla_x n_1| \int_{\sigma_1} p u_1 d\tau_1 \leqslant \kappa_3 \int_{\sigma_1} p t^{-1/2} d\tau_1 \leqslant \kappa_2$$

by (H_1) and boundedness of u_1 and because of the estimates

$$\begin{aligned}
&n_1|_{\operatorname{suppa} p(\cdot,t,\tau_1,\tau_2)} \geqslant \widetilde{n}_2,\\
&|\nabla_x u_1|\big|_{\tau_1 \in \sigma_1} \leqslant \kappa_4 \left\{ \begin{array}{ll} t^{-1/2}, & t \leqslant \tau_1 \\ 1, & t \in (\tau_1, T] \end{array} \right\} \leqslant \kappa_4 t^{-1/2} \quad \text{for } t \in (0,T],
\end{aligned}$$

where $\kappa_2, \kappa_3, \kappa_4$ and $\widetilde{n}_2$ are some positive constants. The estimate for $|\nabla_x u_1|$ follows from (16), (17) and is based on the estimates (see Garoni and Menaldi, 1992; Ladyzhenskaya *et al.*, 1967)

$$\left|\nabla_x \Gamma^{1*}(x,\tau_1;y,0;-\eta)\right| \leqslant c\tau_1^{-(m+1)/2} \exp\left\{-C|x-y|/\tau_1\right\},$$

$$\left|\nabla_x \Gamma^{1}_{*}(x,t;y,0;\eta_1)\right| \leqslant ct^{-(m+1)/2} \exp\left\{-C|x-y|/t\right\},$$

c and C being some positive constants.

Thus $|Y(x,t,\tau_2) - Y(y,t,\tau_2)| \leqslant \kappa|x-y|$ with κ a constant. Hence $Y(x,t,\tau_2)$ is Lipshitz continuous with respect to x in $E^m \times [0,T] \times \overline{\sigma}_2(0)$ (i.e., $Y \in C(E^m \times [0,T] \times \overline{\sigma}_2(0))$, and (H$_4$) shows that $\widetilde{L}_{2*}$ and $\widetilde{L}_2^*$ satisfy all the conditions of Remark 1 in $\left(Q^2_{1*} \cup Q^2_{2*}\right) \cap \omega_1$ and $\left(Q^{2*}_1 \cup Q^{2*}_2\right) \cap \omega_1$, respectively. Therefore we can construct Γ^2_{1*}, Γ^2_{2*}, Γ^{2*}_1, Γ^{2*}_2. Then (18) and (20) for $k=1$ yield u_2 in $\left(Q^2_{1*} \cup Q^{2*}_1\right) \cap \omega_1$, while from (19) and (21) by (22), (27) we get u_2 in $\left(Q^2_{2*} \cup Q^2_{3*}\right) \cap \omega_1$ and $\left(Q^{2*}_2 \cup Q^{2*}_3\right) \cap \omega_1$, respectively. Eq. (20) for $k=4$ gives u_2 in $Q^{2*}_4 \cap \omega_1$.

Let $t \in (T, 2T]$ and assume $\omega_2 = E^m \times [T, 2T] \times \overline{I}$. Knowing u_1 and u_2 for $t \in [0,T]$ by (24), (H$_1$) and because $n_1|_{\mathrm{suppa}\, p(\cdot,t,\tau_1,\tau_2)} \geqslant \widetilde{n}_1$ we get continuous $u_3(x,t,\tau_1,\tau_2,0)$ for $t \in (\tau_3, T]$, then by (23) obtain $u_3(x,t,\tau_1,\tau_2,\tau_3)$ for $t \in (\tau_3, \tau_3+T]$. Observe that, by Remark 1, $|\nabla_x u_3|_{\tau_3=T}|$ for $t > T$ is bounded. From (25), (26), by using (H$_2$) and known continuous $u_3|_{\tau_3=T}$ we get continuous $u_1|_{\tau_1=0}$ and $u_2|_{\tau_2=0}$ too. Then by virtue of (17) with known $u_1|_{\tau_1=0}$ and (20) with known $u_2|_{\tau_2=0}$ we obtain u_1 and u_2 in $Q^{1*} \cap \omega_2$, $Q^{2*}_0 \cap \omega_2$, respectively. Now we can construct Y^*, Y_* and, by the same arguments as before, prove that $\widetilde{L}_2^*$ and $\widetilde{L}_{2*}$ satisfy the conditions of Remark 1. Thus we can obtain Γ^{2*}_1, Γ^{2*}_2 and Γ^2_{2*} (if $\tau_{22} - \tau_{21} > 2T$), which allows us by (20) for $k=1$, (21) for $k=2$, (19) for $k=2$, (21) for $k=3$, (20) for $k=4$, and (18) for $k=4$ to construct u_2 in $\left(\bigcup\limits_{s=1}^{4} Q^{2*}_k \cup Q^2_{2*}\right) \cap \omega_2$.

Proceeding our reasoning we obtain u_1, u_2 and u_3 for $t \in [2T, t^*]$. Restrictions (10) ensure the continuity of u_1, u_2, u_3 across the lines $t=\tau_1$, $t=\tau_2$, $t=\tau_3$, respectively. So Theorem 1 is proved.

COROLLARY 2. Under the hypotheses (H$_3$)–(H$_5$) for $k=1,2$, there exists continuous u_1 and u_2 satisfying problem (1)–(10) in Q^1_* and Q^2_{4*}, respectively. If $\inf\limits_{\overline{Q}_k} \nu_k > 0$ and $u^0_k(x,\tau_k) \to 0$ as $\tau_k \to \infty$, $k=1,2$, then so does $u_k(x,t,\tau_k)$ for $t \leqslant \min(\tau_1,\tau_2)$, $x \in E^m$.

The proof of Corollary follows from the maximum principle (see Friedman, 1968; Garoni and Menaldi, 1992; Ladyzhenskaya *et al.*, 1967).

7. Population Growth and Decay

In this section we consider the case where dispersal moduli a^3_{ij} and a^3_i, $i,j = \overline{1,m}$ are not depending on age τ_1 of the mated male, and prove Theorem 2 and Theorem 3.

Proof of Theorem 2

Set $\widetilde{L}_3(x,t,\tau_2,\tau_3) = \sum_{i,j=1}^{m} a_{ij}^3 \partial^2/\partial x_i \partial x_j + \sum_{i=1}^{m} a_i^3 \partial/\partial x_i$.

Let $\widetilde{\Gamma}_{3*}(x,t;y,\xi;\eta_3,\eta_4)\big($ resp. $\widetilde{\Gamma}_3^*(x,\tau_3;y,\xi;-\eta_4,\eta_3-\eta_4)\big)$ be the fundamental solution of operator $\partial/\partial t - \widetilde{L}_{3*}$ in Q_*^3(resp. $\partial/\partial\tau_3 - \widetilde{L}_{3*}$ in Q^{3*}). Then classic solution of the problem

$$\begin{aligned}
&(\partial/\partial t - \widetilde{L}_3)\widetilde{u}_3 = 0 \quad \text{in } Q_3,\\
&\widetilde{u}_3|_{t=0} = u_3^0 \quad \text{in } E^m \times \sigma_1 \times \sigma_2(\tau_3) \times \sigma_3,\\
&\widetilde{u}_3|_{\tau_3=0} = u_3(x,t,\tau_1,\tau_2,0) \quad \text{in } E^m \times I \times \sigma_1 \times \sigma_2(0)
\end{aligned}$$

with given suitable u_3^0 and $u_3(x,t,\tau_1,\tau_2,0) = pu_1u_2/n_1$ reads

$$\widetilde{u}_3(x,t,\tau_1,\tau_2,\tau_3) = \begin{cases} \int_{E^m} \widetilde{\Gamma}_*^3(x,t;y,0;\tau_2-t,\tau_3-t) \\ \qquad \times u_3^0(y,\tau_1,\tau_2-t,\tau_3-t)\,dy \quad \text{in } Q_*^3, \\ \\ \int_{E^m} \widetilde{\Gamma}^{3*}(x,\tau_3;y,0;t-\tau_3,\tau_2-\tau_3) \\ \qquad \times u_3(y,t-\tau_3,\tau_1,\tau_2-\tau_3,0)dy \quad \text{in } Q^{3*}, \end{cases}$$

and by Lemma 2 verifies $u_3 \leqslant \widetilde{u}_3$ in Q_3. Hence by (22)–(24),

$$\int_{\sigma_1} u_3\,d\tau_1 \leqslant \int_{E^m} dy \widetilde{\Gamma}_*^3(x,t;y,0;\tau_2-t,\tau_3-t) \int_{\sigma_1} u_3^0(y,\tau_1,\tau_2-t,\tau_3-t)\,d\tau_1 \quad \text{in } Q_*^3,$$

$$\begin{aligned}
\int_{\sigma_1} u_3\,d\tau_1 \leqslant \int_{E^m} & dy \widetilde{\Gamma}^{3*}(x,\tau_3;y,0;t-\tau_3,\tau_2-\tau_3) \\
& \times \int_{\sigma_1} \big(pu_1u_2/n_1\big)|_{(y,t-\tau_3,\tau_1,\tau_2-\tau_3)}\,d\tau_1 \quad \text{in } Q^{3*},
\end{aligned}$$

and since

$$\int_{E^m} \widetilde{\Gamma}_*^3(x,t;y,\xi;\tau_2-t,\tau_3-t)\,dy \leqslant 1,$$

$$\int_{E^m} \widetilde{\Gamma}^{3*}(x,\tau_3;y,\xi;t-\tau_3,\tau_2-\tau_3)\,dy \leqslant 1,$$

we have

$$\int_{\sigma_1} u_3\,d\tau_1 \leqslant \begin{cases} \widehat{u}_3 \quad \text{in } Q_*^3, \\ \widehat{p} \sup\limits_{y\in E^m} u_2(y,t-\tau_3,\tau_2-\tau_3) \quad \text{in} \quad Q_*^3 \text{ for } t-\tau_3 \in \overline{I}^*. \end{cases} \tag{28}$$

So assertion (i) is proved.

It remains to prove assertion (ii). We establish inequality (ii) going along the axis t by the step T.

Let $t \in [0, T]$. From (25), (26), by $(28)_1$ with $\tau_3 = T$ we conclude that

$$u_k|_{\tau_k=0} \leqslant \widehat{b}\widehat{u}_3 \leqslant \delta\widehat{u}, \quad k = 1, 2 \quad \text{in } E^m \times [0, T], \tag{29}$$

then from (16)–(21), by (28), (29) and the maximum principle that

$$u_2 \leqslant \delta\widehat{u} \quad \text{for} \quad (x, t, \tau_2) \in E^m \times [0, T] \times \overline{I}, \tag{30}$$

and finally that

$$\int_{\sigma} u_3 \leqslant \widehat{p}\delta\widehat{u} \quad \text{for } x \in E^m,\ t \in (\tau_3, \tau_3 + T],\ \tau_2 \in \sigma_2(\tau_3),\ \tau_3 \in \sigma_3. \tag{31}$$

Let $t \in (T, 2T]$. Estimate (31) with $\tau_3 = T$ and (25), (26) yield

$$u_k|_{\tau_k=0} \leqslant \widehat{b}\widehat{p}\delta\widehat{u} \leqslant \gamma\delta\widehat{u}, \quad k = 1, 2 \quad \text{in } E^m \times (T, 2T]. \tag{32}$$

Then from (18)–(21) by using (32), (31) and the maximum principle we get

$$u_2 \leqslant \gamma\delta\widehat{u} \quad \text{in } E^m \times (T, 2T] \times \overline{I}, \tag{33}$$

which together with (28) yields,

$$\int_{\sigma_1} u_3 \, d\tau_1 \leqslant \widehat{p}\gamma\delta\widehat{u} \quad \text{in } E^m \times (\tau_3, \tau_3 + 2T] \times \sigma_2(\tau_3) \times \sigma_3. \tag{34}$$

Proceeding our argument we prove (ii) for u_2, while this one for u_1 follows from the maximum principle. More rough estimate $u_k \leqslant \delta\widehat{u}\gamma^{t/T}$ in $\overline{Q}_k$, $k = 1, 2$ immediately follows. So Theorem 2 is proved.

Proof of Theorem 3

We consider the case $\tau_{21} > T$. One can analyze the opposite case in the similar way. The assertion (i) is the same as that in Theorem 2. Therefore we have to prove the statement (ii).

Let $t \in [0, T]$. As in the proof of Theorem 2, we obtain

$$\int_{\sigma_1} u_3|_{\tau_3=T} \, d\tau_1 \leqslant \widehat{u}_3 \quad \text{in } E^m \times [0, T] \times \sigma_2(T),$$
$$u_k|_{\tau_k=0} \leqslant \widehat{b}\widehat{u}_3 = q\widehat{u} \quad \text{for } t \in [0, T],$$

$$\begin{aligned} &u_2 \leqslant q\widehat{u} \quad \text{in } E^m \times [0,T] \times [0,t), \\ &u_2 \leqslant \widehat{u} \max\bigl(1, \widehat{u}_3/(\widetilde{\nu}\widehat{u})\bigr) \\ &\quad \leqslant \widehat{u} \max\bigl(1, q/(\widehat{b}\widetilde{\nu})\bigr) \leqslant \widehat{u} \quad \text{in } E^m \times [0,T] \times [t,\infty). \end{aligned} \tag{35}$$

Let $t \in (T, 2T]$. From (i) and (35) it follows that

$$\int_{\sigma_1} u_3|_{\tau_3=T}\, d\tau_1 \leqslant \widehat{p}\widehat{u} \quad \text{in } E^m \times (T,2T] \times \sigma_2(T), \tag{36}$$

which together with (25) and (26) gives

$$u_k|_{\tau_k=0} \leqslant \widehat{b}\widehat{p}\widehat{u} \leqslant q\widehat{u} \quad \text{for} \quad t \in (T, 2T].$$

Hence, by (18)-(21), (36) and because of $\tau_2^2 > 2T$,

$$\begin{aligned} &u_2 \leqslant q\widehat{u} \quad \text{in } E^m \times (T,2T] \times [0,t), \\ &u_2 \leqslant \max\bigl(\widehat{u}, \widehat{u}\widehat{p}/\widetilde{\nu}\bigr) \leqslant \widehat{u}\max\bigl(1, q/\widehat{b}\widetilde{\nu}\bigr) \leqslant \widehat{u} \quad \text{in } E^m \times (T,2T] \times [t,\infty). \end{aligned}$$

In a similar way we obtain

$$\int_{\sigma_1} u_3|_{\tau_3=T}\, d\tau_1 \leqslant \widehat{p}\widehat{u} \quad \text{in } E^m \times [0,\tau_2^2] \times \sigma_2(T), \tag{37}$$

$$u_2 \leqslant \widehat{u} \begin{cases} q & \text{in } E^m \times [0,\tau_2^2] \times [0,t), \\ 1 & \text{in } E^m \times [0,\tau_2^2] \times [t,\infty). \end{cases} \tag{38}$$

Let $t \in (\tau_2^2, \tau_2^4]$. From (25), (26), by (i), one can write

$$u_k|_{\tau_k=0} \leqslant \widehat{p} \int_{\sigma_2(T)} \sup_{\tau_1 \in \sigma_1} b_k \sup_{y \in E^m} u_2(y, t-T, \tau_2 - T)\, d\tau_2,$$

and taking into account (38) get

$$u_k|_{\tau_k=0} \leqslant \widehat{p}\biggl(q\widehat{u} \int_{\tau_2^2}^{t} \sup_{\tau_1 \in \sigma_1} b_k d\tau_2 + \widehat{u} \int_{t}^{\tau_2^4} \sup_{\tau_1 \in \sigma_1} b_k d\tau_2\biggr).$$

Hence

$$u_k|_{\tau_k=0} \leqslant \widehat{p}\widehat{u}\widehat{b} \leqslant q\widehat{u} \quad \text{in } E^m \times (\tau_2^2, \tau_2^4], \quad k = 1, 2.) \tag{39}$$

From (20), by (38), (39), it follows that

$$u_2 \leqslant q\widehat{u} \quad \text{in } E^m \times (\tau_2^2, \tau_2^4] \times [0, \tau_2^2]. \tag{40}$$

Eqs. (19) (in the case $\tau_2^3 > 2\tau_2^2$) and (21), estimates (i) with $\tau_3 = T$, (38), (40), and the maximum principle show that

$$u_2 \leqslant \widehat{u} \begin{cases} q & \text{in } E^m \times (\tau_2^2, \tau_2^4] \times [0, t), \\ 1 & \text{in } E^m \times (\tau_2^2, \tau_2^4] \times [t, \infty). \end{cases}$$

Let $t \in (\tau_2^4, 2\tau_2^4]$. Reasoning as above we obtain

$$u_k|_{\tau_k=0} \leqslant \widehat{pb}q\widehat{u} \leqslant q^2\widehat{u} \quad \text{in } E^m \times (\tau_2^4, 2\tau_2^4],$$

and

$$u_2 \leqslant \widehat{u} \begin{cases} q^2 & \text{in } E^m \times (\tau_2^4, 2\tau_2^4] \times [0, t - \tau_2^4), \\ q & \text{in } E^m \times (\tau_2^4, 2\tau_2^4] \times [t - \tau_2^4, t), \\ 1 & \text{in } E^m \times (\tau_2^4, 2\tau_2^4] \times [t, \infty). \end{cases}$$

Continuing our argument we prove the assertion (ii) for u_2, then, from (16), (17) and the maximum principle, statement (ii) follows for u_1. This ends the proof.

8. One More Estimate for u_1 and u_2 Growth.

In this section we obtain upper estimates for u_1, u_2 and $\int_{\sigma_1} u_3\, d\tau_1$ based on the population intrinsic growth rate λ_0. Under the hypotheses of Theorem 2 by the comparison principle we can prove that u_1, u_2 and $\int_{\sigma_1} u_3\, d\tau_1$ possess majorants $U(t, \tau_1)$, $U(t, \tau_2)$ and $U_3(t, \tau_2, \tau_3)$, respectively, satisfying the following problem

$$\begin{aligned}
&(D_1 - \nu_{1*})U_1 = 0 \quad \text{in } I \times I, \\
&(D_2 - \nu_{2*})U_2 = -U_2 \begin{cases} 0, & \tau_2 \notin \sigma_2(0), \\ p_*, & \tau_2 \in \sigma_2(0) \end{cases} \\
&\qquad\qquad + \begin{cases} 0, & \tau_2 \notin \sigma_2(T), \\ U_3|_{\tau_3=T}, & \tau_2 \in \sigma_2(T) \end{cases} \quad \text{in } I \times (I \setminus \bigcup_{s=1}^{4} \tau_2^s), \\
&(D_3 - \nu_{3*})U_3 = 0 \quad \text{in } I \times \sigma_2(\tau_3) \times \sigma_3, \\
&U_1|_{t=0} = U_1^0, \quad U_2|_{t=0} = U_2^0, \quad U_3|_{t=0} = U_3^0, \\
&U_1|_{\tau_1=0} = \int_{\sigma_2(T)} b_1^* U_3|_{\tau_3=T}\, d\tau_2, \quad U_2|_{\tau_2=0} = \int_{\sigma_2(T)} b_2^* U_3|_{\tau_3=T}\, d\tau_2, \\
&U_3|_{\tau_3=0} = p^* U_2, \quad [U_2|_{\tau_2=\tau_2^s}] = 0, \quad s = \overline{1,4},
\end{aligned}$$

where:

$$\nu_{1*}(\tau_1) = \inf_{(x,t)\in E^m\times\overline{I}} \nu_1, \quad \nu_{2*}(\tau_2) = \inf_{(x,t)\in E^m\times\overline{I}} \nu_2,$$

$$\nu_{3*}(\tau_2,\tau_3)=\inf_{(x,t,\tau_1)\in E^m\times\overline{I}\times\overline{\sigma}_1}\nu_3,$$
$$U_1^0(\tau_1)=\sup_{x\in E^m}u_1^0,\quad U_2^0(\tau_2)=\sup_{x\in E^m}u_2^0,\quad U_3^0(\tau_2,\tau_3)=\sup_{x\in E^m}\int_{\sigma_1}u_3^0\,d\tau_1,$$
$$b_1^*(\tau_2)=\sup_{(x,t,\tau_1)\in E^m\times\overline{I}\times\overline{\sigma}_1}b_1,\quad b_2^*(\tau_2)=\sup_{(x,t,\tau_1)\in E^m\times\overline{I}\times\overline{\sigma}_1}b_2,$$
$$p_*(\tau_2)=\inf_{(x,t,\tau_1)\in E^m\times\overline{I}\times\overline{\sigma}_1}p,\quad p^*(\tau_2)=\sup_{(x,t,\tau_1)\in E^m\times\overline{I}\times\overline{\sigma}_1}p.$$

In (Skakauskas, 1997) we have constructed the following largetime $(\max(\tau_1,\tau_2)<t)$ asymptotic behavior of U_1, U_2 and U_3:

$$U_1\sim c_1\exp\{\lambda_0(t-\tau_1)-\int_0^{\tau_1}\nu_{1*}(\eta)d\eta\},$$
$$U_2\sim c_2 f_2(\tau_2)\exp\{\lambda_0(t-\tau_1)\},$$
$$U_3\sim c_2p^*(\tau_2-\tau_3)f_2(\tau_2-\tau_3)\exp\{\lambda_0(t-\tau_2)-\int_0^{\tau_3}\nu_{3*}(\eta+\tau_2-\tau_3,\eta)\,d\eta\},$$

where c_1, c_2 are two positive constants, λ_0 is a unique real root of the characteristic equation

$$\int_{\sigma_2(0)}b_2^*(\eta+T)g(\eta)f_2(\eta)\exp\{-\lambda\eta\}\,d\eta=1,$$

and $f_2(\tau_2)$ satisfies the following equation

$$\Big(\frac{d}{d\tau_2}-\nu_{2*}\Big)f_2=-\begin{cases}0, & \tau_2\notin\sigma_2(0),\\ p_*, & \tau_2\in\sigma_2(0)\end{cases}$$
$$+\begin{cases}0, & \tau_2\notin\sigma_2(T),\ f_2(0)=1,\ [f_2(\tau_2^i)]=0,\ i=1,4,\\ g(\tau_2)f_2(\tau_2-T), & \tau_2\in\sigma_2(T),\ [f_2(\tau_2^i)]=0,\ i=2,3\end{cases}$$

with $g(\tau_2)=p^*(\tau_2-T)\exp\big\{-\int_0^T\nu_{3*}(\eta+\tau_2-T,\eta)\,d\eta\big\}$.
Clearly, u_1, u_2 vanish as t increases and $\lambda_0<0$.

References

Friedman, A. (1968). *Differential Equations with Partial Derivatives of the Parabolic Type*. Mir, Moscow (in Russian).

Garoni, M.G., and J.L. Menaldi (1992). Green functions for second order parabolic integro-differential problems. *Pitman Research Notes in Mathematics*, Series 275 ISSN 0269–3677, New York.

Ladyzhenskaya, O.A., V.A. Solonnikov, N.N. Ural'tseva (1967). *Linear and Quasilinear Equations of Parabolic Type*. Nauka, Moscow (in Russian).

Skakauskas V. (1994). An evolution model of an autosomal polylocal polyallelic diploid population taking into account crossing-over and gestation period. *Lith. Math. J.*, **34**(3), 288–300.

Skakauskas, V. (1995). Evolution of migrating two-sexes populations. *Informatica*, **7**(1), 83–96.
Skakauskas, V. (1997). On solvability of some models of migrating populations. *Lith. Math. J.*, **37**(4), 426–442.
Skakauskas, V. (1997). On asymptotics of a population model with random mating, Mathematical modeling and complex analysis. In R. Ciegis (Ed.), *Proceedings of the Second International Conference "Mathematical modeling and complex analysis"*, Vilnius, 143–151.

V. Skakauskas has graduated from Leningrad University (Faculty of Mathematics and Mechanics) in 1965. He received Ph.D. degree from the Leningrad University in 1971. He is an Associate Professor and a head of the Department of Differential Equations and Numerical Analysis at Vilnius University. His research interests include modelling in fluids and gas mechanics, physics, ecology and genetics.

Vieno migruojančios populiacijos amžiaus ir lyčių struktūros dinamikos modelio matematinė analizė, įskaitant atsitiktinį kryžminimąsi ir patelių nėštumą

Vladas SKAKAUSKAS

Tiriamas vieno migruojančios populiacijos amžiaus ir lyčių struktūros dinamikos modelio klasikinis išspendžiamumas. Populiaciją sudaro patinėliai, neapvaisintos ir apvaisintos patelės. Reproduktyvieji intervalai ir nėštumo trukmė laikomi fiksuotais, o kryžminimosi sistema, nesudarant pastoviųjų vedybinių porų, yra atsitiktinė. Individų migracijos mechanizmas aprašomas bendruoju tiesiniu tolygiai elipsinių 2-osios eilės operatoriumi daugiamatėje erdvėje. Įrodytas klasikinio sprendinio egzistavimas ir vienatis.