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Simulation of Wet Film Evolution and the Euclidean Steiner Problem

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Abstract. The Steiner problem asks for the shortest network that spans a given set of fixed points in the Euclidean plane. The problem is NP hard.

The result of simulation of an idealized "wet" film connecting fixed points is a length-minimizing curve. Increasing the exterior pressure step by step we are able to achieve the film configuration near to the Steiner minimal tree. "Dead-point" situations may occur for some symmetric allocation of fixed points.

The limited simulation experiments show that the average computation time depends almost linearly on the number of fixed points for the situations without "dead-points".

Key words: optimization, Steiner problem, soap film, simulation, instability.

1. Introduction

Minimizing a network length is one of the oldest optimization problems. In the seventeenth century the following problem was suggested: find the point P that minimizes the sum of the distances from P to each of the three given points in the plane.

Fermat, Torricelli, and Cavalieri independently derived solutions to this problem. P may be inside the triangle formed by the points; the angles formed by the lines connecting P to each of the points are all $120°$. P may also be one of the vertices, and the angle formed by connecting P to the other vertices is greater than or equal to $120°$.

Fig. 1 illustrates the method. To calculate P for points A, B, and C, first construct an equilateral triangle ACX using the longest edge AC of the triangle formed by the three original points. Circumscribe a circle about the equilateral triangle, and construct a line from the third point B to the far vertex X of the equilateral triangle. The point at which the line and the circle intersect is P.

In the nineteenth century Jacob Steiner expanded the problem to an arbitrarily large set of points. He involved only one point still, forming a star-like shape when P was joined to each of the points. Later the problem was extended even further by allowing the addition of an arbitrary number of points. Courant and Robbins (1941) popularised this problem, which became known as the Steiner Minimal Tree problem. Each point added to create a minimal network is called a Steiner Point.

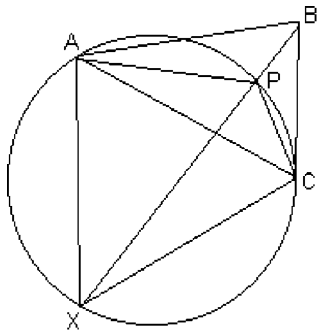

Fig. 1. Illustration of the method used to construct the point P.

The Steiner problem asks for the shortest network that spans a given set of n fixed points in the Euclidean plane. Let T be a minimal network consisting of vertices and edges connecting the points, and let $G(T)$ be the graph of T. It is not difficult to show that T must have the following properties (Gilbert and Pollak, 1966):

- all edges are straight lines;
- $G(T)$ is a tree;
- the angle between any two edges meeting at a vertex is at least 120 degrees;
- all Steiner points have degree 3 and the edges meeting at a Steiner point make angles of precisely 120 degrees with each other;
- the number of Steiner points is at most $n - 2$.

Any network with these properties is defined as a Steiner tree. However, the properties are not sufficient that the length of the Steiner tree T be minimal. The number of different topologies grows exponentially as n increases. It has been shown that the Euclidean Steiner problem is NP hard (Garey, Graham, and Johnson, 1977). There can be no polynomial-time algorithm for generating minimal Steiner trees.

The Steiner problem can be seen in nature. Place pins to represent points into a flat surface and place a flat surface on the other ends of the pins. Dip this apparatus into a soap solution and remove. The soap film formed between the pins (points) will minimize the overall surface area. Since the distance between the surfaces is constant, the film will also minimize the total distance forming Steiner points in the process (see Fig. 2).

We can find an attempt to simulate the evolution of an idealized soap film in order to solve the Steiner problem in (Jakutaviciene, 1965). However, the results of these computational experiments were not published.

2. Simulation of Wet Film Evolution

Idealized infinitely thin soap films of this sort are called “dry” (Brakke and Morgan, 1997). We can get not one stable “dry” film configuration of different total length for the same points. For example, the configuration in Fig. 2, which may be denoted as horizontal double Y, is not unique; the vertical double Y configuration is also stable.

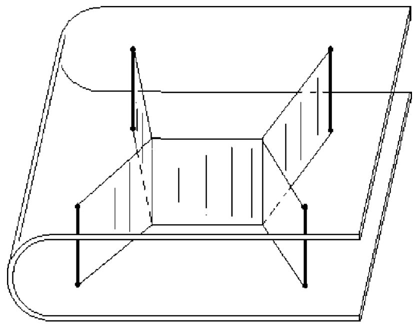

Fig. 2. Illustration of a soap film, forming Steiner points.

The main idea is to use the mathematical model for an idealized "wet" one-dimensional soap film, connecting the fixed points with some liquid inside the film. The wet film tries to shrink to the minimum length, subject to a constraint. The constraint can be either that the liquid has a fixed interior area, or that the liquid be at a fixed pressure. The fixed pressure constraint is appropriate if the wet film is meant to be a cross-section of a larger three-dimensional wet foam. The theorems in (Hass and Morgan, 1996) and (Brakke and Morgan, 1997) state that a length-minimizing curve enclosing a region of a fixed area to be composed of:

- circular arcs of equal positive outward curvature
  and
- straightline segments multiplicity two.

The radius R of arcs depends on the external pressure p:

$$p = 1/R.$$

Fig. 3 illustrates the shapes of a wet film connecting four points for increasing pressures (and various decreasing radii R). The centers of arcs are also indicated in the figure.

The illustration is useful to substantiate the idea that step by step reducing radius R (or equivalently increasing the pressure) of the length-minimizing curve we are able to achieve a configuration of the length-minimizing curve close to the Steiner minimal tree.

2.1. *Definitions*

Let us introduce definitions for some characteristic points of the length-minimizing curve (see Fig. 4).

Points that must be connected will be referred to as *fixed points* P_i $(i = 1, \ldots, n)$.

We define the corners of the length-minimizing curve interior as *corners* C_j $(j = 1, \ldots, m)$. The corners are *wet* if their coordinates are the same as the coordinates of the corresponding fixed points (Fig. 4a); otherwise, they are *dry* (Fig. 4b). The maximal number m of corners may amount to $3(n-2)$.

Each corner C_j has its *base*. A base may be:

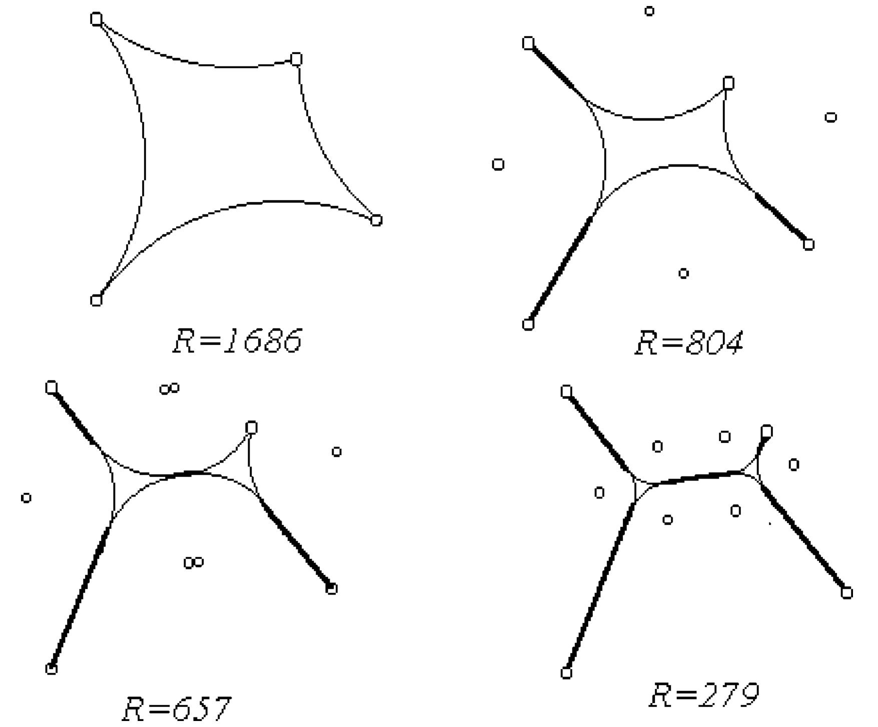

Fig. 3. Wet film connecting four points under various pressures.

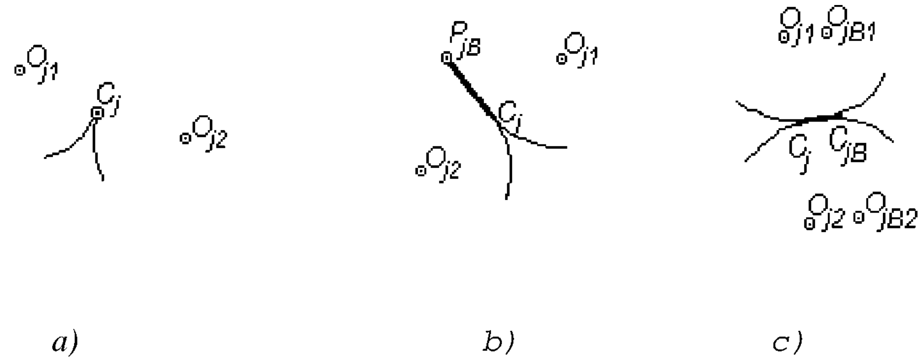

Fig. 4. Characteristic points of the length-minimizing curve.

- a fixed point directly connected by the help of a straightline segment to the corner C_j (the fixed point P_{jB} in Fig. 4b) or
- another corner directly connected by the help of a straightline segment to the corner C_j (the corner C_{jB} in Fig. 4c).

The wet corner and its base have the same coordinates.
Each corner C_j has two centers of its arcs: O_{j1} and O_{j2} (Fig. 4a and 4b).

2.2. *A System of Equations*

Let us denote an Euclidean distance between two points A_i and A_j as

$$d(A_i, A_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2},$$

where x_i, y_i, x_j, y_j are coordinates of the points A_i and A_j.

Let radius R and fixed points P_i $(i = 1, \dots, n)$ be given. Then the coordinates of centers and corners of the length-minimizing curve satisfy the following system of nonlinear equations of three types.

1. For each wet corner C_j the base P_{jB} of which has the same coordinates we have the following two equations (see Fig. 4a):

$$d(P_{jB}, O_{j1}) = R; \quad (1)$$
$$d(P_{jB}, O_{j2}) = R. \quad (2)$$

2. For each dry corner C_j the base of which is a fixed point P_{jB} we have the following two equations (see Fig. 4b):

$$d(O_{j1}, O_{j2}) = 2R; \quad (3)$$
$$d(P_{jB}, O_{j1}) = d(P_{jB}, O_{j2}). \quad (4)$$

3. For each dry corner C_j whose base is another corner C_{jB} we have the following two equations (see Fig. 4c):

$$d(O_{j1}, O_{j2}) = 2R; \quad (5)$$
$$d(C_{jB}, O_{j1}) = d(C_{jB}, O_{j2}). \quad (6)$$

The coordinates of dry corners C_{jB}: $x(C_{jB})$ and $y(C_{jB})$ in equation (6) may be simply calculated from the coordinates of two corresponding centers O_{jB1}, O_{jB2} (the corners are located in the middle of centers):

$$x(C_{jB}) = \frac{x(O_{jB1}) + x(O_{jB2})}{2},$$
$$y(C_{jB}) = \frac{y(O_{jB1}) + y(O_{jB2})}{2}.$$

Unknowns of the system are the coordinates of centers O.

The number of unknowns is equal to the number of equations because each center belongs to two corners (see Fig. 3).

2.3. *Initial Stage of Simulation*

The simulation of film evolution begins with extremely low values of pressure. Usually, the initial values of radius R are defined two or three times higher than the longest distance between the fixed points. Only a part of the fixed points belonging to a convex hull of fixed points is included into the length-minimizing curve.

All corners are wet at the initial stage of simulation, therefore the system consists of equations only of type (1) and (2). In this case, the unknowns may be simply solved in an explicit way, because the system may be divided into systems of equation pairs.

The values of the unknowns obtained are used as initial approximations for the next step of simulation with a decreased radius R.

2.4. *Steps of Simulation*

The steepest descent method (Burden and Faires, 1993) is used for solving the system of nonlinear equations in the general case. The results of simulation of the previous step are used as initial approximations. The increments of radius ΔR from step to step are small, therefore the number of iterations for sufficient accuracy is extremely small.

At each step of decreasing R we check the occurrence of next events:

- a wet corner became dry;
- a pair of opposite arcs touched each other;
- a fixed point not connected so far was touched by the length-minimizing curve.

The respective corrections must be carried out in the equation system depending on the event.

When a wet corner becomes dry its two equations of type (1)–(2) must be changed to type (3)–(4).

When a pair of opposite arcs touches each other, two new dry corners similar to that in Fig. 4c are born. Four new equations of type (5)–(6) must be included into the equation system.

When a fixed point not connected so far is touched by the length-minimizing curve, a new wet corner is born. A pair of new equations of type (1)–(2) must be included.

The simulation process stops when radius R achieves low values.

The computer time of the simulation process depends on the number of fixed points n and their allocation. Usually, the time for PC 486 does not exceed one minute for 8 fixed points.

3. Illustration of the Simulation Process

Fig. 5 illustrates five simulation stages of evolution for seven fixed points.

In the beginning (Fig. 5a) the convex hull includes only six fixed points. Fig. 5b shows the step when a point not connected was touched by the length-minimizing curve.

The pair of opposite arcs touched each other in Fig. 5c.

Another contact of the opposite arcs is illustrated in Fig. 5d.

The final phase is shown in Fig. 5e. We can evidently see the Steiner tree configuration because the radius of arcs is small enough.

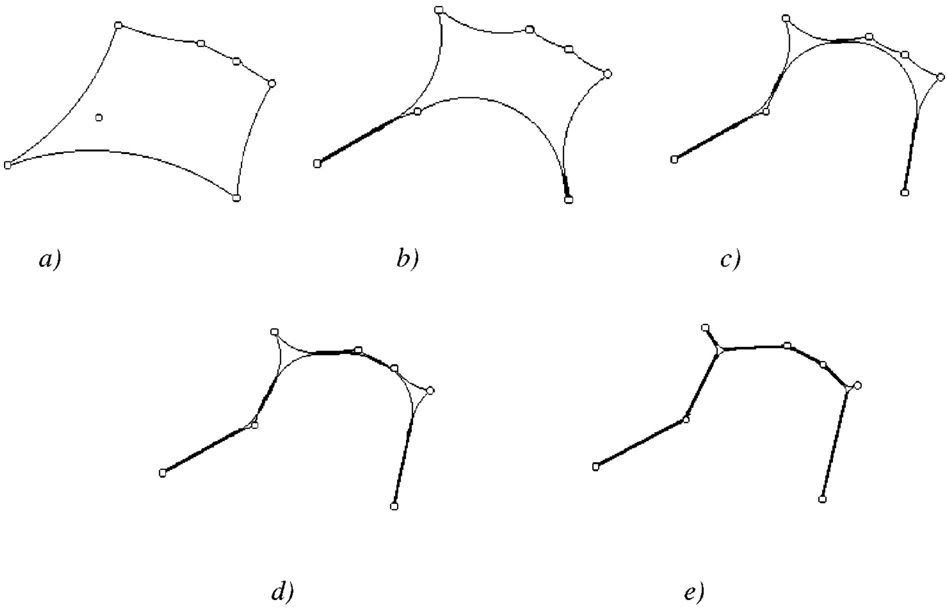

Fig. 5. Five stages of evolution.

4. Dead-points in the Simulation Process

The results of simulation in each step of reducing radius R are used as initial approximations for the next step of simulation. The process of simulation is successful if small changes of radius R in each step lead to a nearby stable film curve. However for some symmetric allocation of fixed points at specific stages of simulation, some “dead-point” situations may occur. In such a situation the solution of equations of (1)–(6) type can not be found from the initial approximation of the previous step. The situation is rare for a small number n of fixed points but it occurs more frequently for larger n.

Theoretical investigations of possible stable wet film curves are carried out only for a simple situation when fixed points are located at four corners of a rectangle (Brakke and Morgan, 1997). They are useful in understanding and overcoming a dead-point phenomenon.

4.1. *Stability Diagram for a Rectangular Wet Film*

In this chapter, we use the stability diagram (Fig. 6) for four points fixed on the symmetric coordinates $(\pm x, \pm y)$ with the pressure fixed at 1 (Brakke and Morgan, 1997). The diagram shows what stable configurations of a film are in various locations of fixed points and pressure.

Area J of the diagram is for configurations when all the corners are wet. The pressure is low and the points are located relatively symmetrically. The situations occur at the initial steps of evolution.

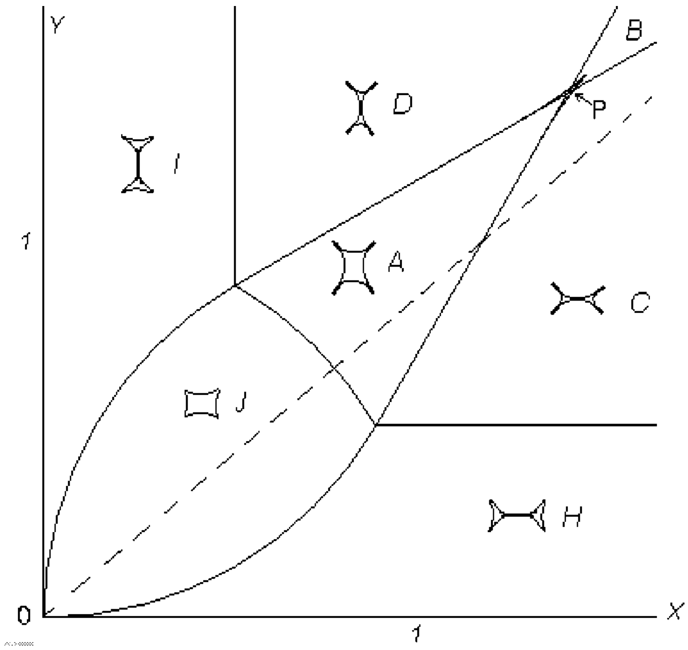

Fig. 6. Stability diagram for a rectangular situation under constant pressure.

Areas I and H are for configurations when due to an extremely asymmetric situation of fixed points the opposite sides touch each other. Area I is for a vertical orientation of configuration, and H is for a horizontal one.

Areas C and D are similar to I and H except that all the corners are dry and the pressure is higher.

Area A is for relatively symmetric situations. The corners are dry and the opposite sides do not touch each other yet.

Area B is for the final steps of evolution: the pressure is high, radius R is small and the opposite sides touch each other. The orientation (vertical or horizontal) depends on the orientation in previous steps.

A relatively small area P is for dead-point situations. When the film configuration evolves due to increasing pressure from area A to P, no nearby stable configurations can be found, and we get into a dead-point situation.

The dotted line in the figure shows one of possible film evolution trajectories, where $x/y = \text{const}$; $R/x \to 0$. We can see three phases:

- in the area J all corners are wet;
- in the area A all corners are dry;
- in the area C we have a horizontal double Y configuration.

Analogous questions about stability diagrams in the general case remain controversial.

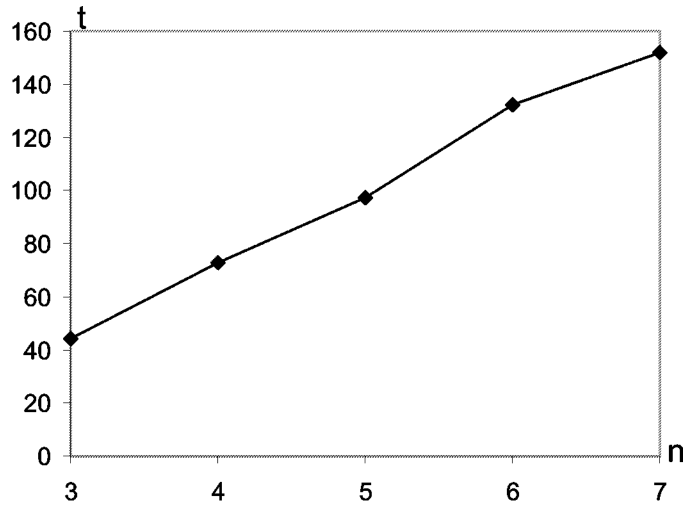

Fig. 7. Average computation time t versus the number n of fixed points.

5. Experiments and Analysis

In order to examine the algorithm implemented, it is interesting to view at the computation time t as a function of the number n of fixed points. For this reason we generate multiple random data sets for a varying number of points and run the algorithm on each data set in turn. The graph of the average computation time t versus the number n of points has been illustrated in Fig. 7. The trendline shown is nearly linear. This is significant, because the Steiner problem cannot truly be solved in polynomial time.

6. Conclusions

The investigations show that simulation of wet film evolution is an interesting tool for solving the Steiner problem. Further investigations may be directed to overcome difficulties due to the dead-point problem.

It is obvious that the approach may be used in solving other combinatorial problems related with the shortest length, for example, the Travelling Salesman Problem.

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Drėgnų plėvelių evoliucijos modeliavimas ir Euklidinis Šteinerio uždavinys

Vydūnas ŠALTENIS

Šteinerio uždavinys siekia rasti trumpiausio ilgio tinklą, jungiantį fiksuotų taškų aibę plokštumoje. Uždavinys yra NP sunkus.

Modeliuodami idealizuotos drėgnos plėvelės, jungiančios fiksuotus taškus, formą gauname minimalaus ilgio kreivę. Palaipsniui didinant išorinį slėgį plėvelės forma palaipsniui artėja prie Šteinerio minimalaus medžio. Modeliavimo eksperimentai rodo, kad kai kuriems simetriškiems fiksuotų taškų išsidėstymams toks evoliucijos procesas gali įstrigti mirties taškuose.

Ribotos apimties eksperimentų modeliavimo laiko vidurkis beveik tiesiškai priklauso nuo fiksuoto taškų skaičiaus situacijoms be mirties taškų.