133

Optimal Control of a Well-Stirred Bioreactor in the Presence of Stochastic Perturbations

Vadim AZHMYAKOV

Institute of Mathematics and Computer Sciences, Ernst-Moritz-Arndt University of Greifswald
Jahnstr. 15a, D–17487 Greifswald, Germany
e-mail: azmjakow@uni-greifswald.de

Abstract. We study the stochastic model for bioremediation in a bioreactor with ideal mixing. The dynamics of the examined system is described by stochastic differential equations. We consider an optimal control problem with quadratic costs functional for the linearized model of a well-stirred bioreactor. The optimal control is based on the optimal robust state estimates. The approximate optimal solution is obtained as a linear feedback.

Key words: optimal robust estimation, optimal control, Bellman optimality principle, linear feedback.

1. Introduction

The estimation theory and optimal control of stochastic systems are the important techniques of the population modeling (see Clark, 1990; Heinricher *et al.*, 1995; Bernard and Guy, 1988; Lenhart *et al.*, 2001). In this paper we investigate the problem of computing the optimal control in a linearized stochastic model based on robust recursive estimates. We consider a setting of an industrial bioreactor (for example drug production or sewage treatment) as a controllable process (Bailey and Ollis, 1986; Heinricher *et al.*, (1995); Dochainand Bastin, 1990).

An bioreactor is a bioremediation system containing bacteria in the presence of a contaminant. Assume that bacteria and contaminant are specified by corresponding spatially uniform concentrations. According to A. Heinricher *et al.* (1995) we deal with a well-stirred bioreactor (bioreactor with ideal mixing) described by two ordinary differential equations

$$\frac{d}{d\tau}\xi_1(\tau) = q(u)\xi_1(\tau) - D\xi_1^m(\tau) \quad \text{a.e. } \tau \in [0, T], \tag{1}$$

$$\frac{d}{d\tau}\xi_2(\tau) = -K\xi_2(\tau)\xi_1(\tau) \quad \text{a.e. } \tau \in [0, T], \tag{2}$$

$$\xi_1(0) = \xi_1^0 > 0, \quad \xi_2(0) = \xi_2^0 > 0, \tag{3}$$

where $\xi_1(\tau) \in \mathbb{R}_+$ is the concentration of bacteria, $\xi_2(\tau) \in \mathbb{R}_+$ is the concentration of some contaminant, $0 \leqslant u(\tau) \leqslant M$, $M \in \mathbb{R}_+$ is the spatially uniform concentration of

a nutrient, D, m, K are known positive constants and a.e. means almost everywhere. The function $u(\cdot)$ plays the role of a control. We have a control process governed by equations (1) and (2) for $0 < \tau < T$. By ξ_1^0 and ξ_2^0 we denote here respectively initial concentrations of bacteria and contaminant. Let $q(\cdot)$ be an increasing function. In the model (1)–(3) the bacteria growth rate is given by $q(u)\xi_1$. This growth rate results from the Michaelis–Menton formula for enzyme kinetics (see, for example, Bailey and Ollis, 1986). According to Heinricher *et al.* (1995) the bacteria death rate is equal to $D\xi_1^m$. Note that in the model (1)–(3) the contaminant is not needed for growth.

For the process (1)–(3) the following optimal control problem is examined in Heinricher *et al.* (1995)

$$\text{minimize } J(u) := \int_0^T [\Lambda \xi_2^\alpha(\tau) + \Psi u^\gamma(\tau) - \Gamma \xi_1^\delta(\tau)]d\tau \qquad (4)$$
$$\text{subject to } (1) - (3),$$

where $u(\cdot)$ is the measurable control function such that $0 \leqslant u(\tau) \leqslant M$, the constants $\Lambda, \Psi, \Gamma, \alpha, \delta$ are nonnegative and $\gamma \geqslant 1$. The objective functional will combine the goals of degrading the contaminant, not overusing the nutrient, and maximizing bacterial colony size.

It will readily be seen that we are concerned with the control system (1)–(3). Real control systems in the bioscience always function under conditions of incomplete information. This effect is caused by distinct perturbations in the biosystem. In the case of well-stirred bioreactor we deal with the disturbances of time-varying concentrations of bacteria and contaminant. Therefore we extend the model of bioreactor (1)–(3) and introduce the following system of stochastic differential equations

$$dY(\tau) = \tilde{a}(Y(\cdot), u(\tau))d\tau + b(\tau)dw(\tau), \qquad (5)$$
$$Y(0) = \xi^0,$$

where $Y(t) := (Y_1(t), Y_2(t))'$ is the state vector, $\tilde{a}: \mathbb{C}([0,T]) \times U \to \mathbb{R}^2$, and $b: [0,T] \to \mathbb{R}^{2\times 2}$, $U := [0, M]$. By $w(\tau) = (w_1(\tau), w_2(\tau))'$ we denote here a Wiener process and $\xi^0 = (\xi_1^0, \xi_2^0)'$. The admissible controls $u(\cdot)$ are measurable functions such that $0 \leqslant u(\tau) \leqslant M$. We assume that

$$\tilde{a}(Y(\cdot), u(\tau)) = \tilde{a}(Y(\tau), u(\tau)),$$

with

$$\tilde{a}(Y(\tau), u(\tau)) = \begin{pmatrix} q(u(\tau))Y_1(\tau) - DY_1^m(\tau) \\ -KY_2(\tau)Y_1(\tau) \end{pmatrix}.$$

Let $b(\cdot)$ be a continuous function. The matrix $b(\tau)$ has the following structure

$$b(\tau) = \begin{pmatrix} b_{11}(\tau) & 0 \\ 0 & b_{22}(\tau) \end{pmatrix}.$$

The equation (5) is considered on the probability space $(\mathbb{C}([0, T], \mathcal{B}, P))$, where P is the measure appropriate to the stochastic differential equation

$$d\zeta(\tau) = b(\tau)dw(\tau).$$

The (stochastic) state vector $Y(\tau) = (Y_1(\tau), Y_2(\tau))'$ is the vector of the concentrations of bacteria and of contaminant.

It is well known that linearization is an important tool for numerical methods of optimal control. Let $q(\cdot)$ be a continuously differentiable function. Using (5) we evaluate the linearized model of the bioreactor

$$dY(\tau) = (a_1(\tau)Y(\tau) + a_2(\tau)u(\tau))dt + b(\tau)dw(\tau), \tag{6}$$
$$Y(0) = \xi^0, \tag{7}$$

where

$$a_1(\tau) := \frac{\partial}{\partial Y}\tilde{a}(\xi^*(\tau), u^*(\tau)) = \begin{pmatrix} q(u^*(\tau)) - mD(\xi_1^*)^{m-1}(\tau) & 0 \\ -K\xi_2^*(\tau) & -K\xi_1^*(\tau) \end{pmatrix},$$

and

$$a_2(\tau) := \frac{\partial}{\partial u}\tilde{a}(\xi^*(\tau), u^*(\tau)) = \begin{pmatrix} \frac{d}{d\tau}q(u^*(\tau)) \\ 0 \end{pmatrix}.$$

The pair $(\xi^*(\cdot), u^*(\cdot))$, with $\xi^*(\tau) := (\xi_1^*(\tau), \xi_2^*(\tau))'$, is an optimal solution of the deterministic optimal control problem (4) (see Heinricher *et al.*, 1995). The equation (6) is considered on the same probability space $(\mathbb{C}([0, T], \mathcal{B}, P)$. According to the general existence results (Liptser and Shiryayev, 1977) we obtain the existence of the unique continuous solution $Y(\cdot)$ of the equation (6) for every admissible control function $u(\cdot)$.

For the linear model (6)–(7) we formulate the following stochastic optimal control problem with quadratic costs functional

$$\text{minimize } J(u) = \frac{1}{2}\text{E}\Big(\int_0^T [(Y'(\tau), u(\tau))\Theta(\tau)(Y'(\tau), u(\tau))']d\tau\Big) \tag{8}$$
$$\text{subject to (6)} - \text{(7)}, \tag{9}$$

where $u(\cdot)$ is the measurable control function such that $0 \leqslant u(\tau) \leqslant M$ and $\Theta(\tau) \in \mathbb{R}^{3\times 3}$. Now we present the following existence theorem.

Theorem 1. *Let $q(\cdot)$ be a continuously differentiable function and $b(\cdot)$ be continuous. Then there exists an optimal solution $u(\cdot)$ of (8)–(9).*

The proof is found in Liptser and Shiryayev (1977). The Bellman's equation for the problem (8)–(9) may be written as

$$\min_{u \in U}\Big[\frac{\partial}{\partial\tau}\mathcal{S}(\tau, \xi) + \langle a_1(\tau)\xi + a_2(\tau)u, \frac{\partial}{\partial\xi}\rangle\mathcal{S}(\tau, \xi)$$

$$+ \frac{1}{2} \Big(b_{11}(\tau) \frac{\partial}{\partial \xi_1} + b_{22}(\tau) \frac{\partial}{\partial \xi_2} \Big)^2 \mathcal{S}(\tau, \xi) + (\xi'(\tau), u(\tau)) \, \Theta(\tau) \, (\xi'(\tau), u(\tau))' \Big]$$

where $\xi(\tau) = (\xi_1(\tau), \xi_2(\tau))'$ and $\mathcal{S}(\cdot, \cdot)$ is the Bellman function.

The remainder of the paper is organized as follows. In Section 2 we deal with discretized optimal control problem in presence of non-Gaussian perturbations. In Section 3 we formulate an auxiliary optimization problem for computing an optimal robust estimation and obtain a new minimax robust estimator of the Kalman type. In the fourth Section we establish the conditional normality of innovations. In the fifth Section we present a solution of the discrete optimal control problem based on the obtained estimation and on the Bellman's optimality principle.

2. Problem Formulation

In this section we introduce a discrete-time approximating problem to the continuous-time optimal control problem (8)–(9). We consider the approximating problem under conditions of incomplete information and extend the model under the assumption that the state vector is observed.

Let N be a sufficiently large positive integer number and $\Delta \tau := T/N$ be the fixed step size. We define the equidistant mesh $\{\tau_0, ..., \tau_N\}$

$$\tau_0 < ... < \tau_N, \quad \tau_t = t \Delta \tau,$$

where $t = 0, ..., N$ is the "discrete time". Denote $\tilde{\beta}_t := Y(\tau_t)$, $u_t := u(\tau_t)$. Using Euler-type discretizations one can deduce the approximating state equation

$$\tilde{\beta}_t = \tilde{F}_t \tilde{\beta}_{t-1} + \tilde{H}_t u_t + \tilde{v}_t, \tag{10}$$

$$\tilde{\beta}_0 = \xi^0, \quad t = 1, ..., N, \tag{11}$$

where $\tilde{F}_t := (I - \Delta \tau a_1(\tau_t))^{-1}$, $\tilde{H}_t := (I - \Delta \tau a_1(\tau_t))^{-1} a_2(\tau_t)$. We assume that the corresponding matrices are invertible. By $\tilde{v}_t$ we denote here independent, identically distributed Gaussian random variables with values in $\mathbb{R}^2$ such that

$$\mathrm{E}(\tilde{v}_t) = 0, \ \mathrm{E}(\tilde{v}_t \tilde{v}'_s) = \tilde{Q}_t \delta_{ts}, \quad t, s = 1, ..., N \ ,$$
$$\tilde{v}_t \sim \mathcal{N}(\mathrm{E}(\tilde{v}_t), \mathrm{E}(\tilde{v}_t \tilde{v}'_s)),$$

where δ_{ts} is the Kronecker delta and $\tilde{Q} := b^2(\tau_t)$.

In parallel with the state equation (10) we consider the linear observation model for the state vector $\tilde{\beta}$ in (10)

$$\tilde{y}_t = \tilde{Z}_t \tilde{\beta}_t + \tilde{\epsilon}_t, \quad t = 0, ..., N \tag{12}$$

where $\tilde{Z}_t \in \mathbb{R}^{2 \times 2} \ \forall t = 1, ..., N$, and

$$\mathrm{E}(\tilde{\epsilon}_t) = 0, \ \mathrm{E}(\tilde{\epsilon}_t \tilde{\epsilon}'_s) = \tilde{V}_t \delta_{ts}, \quad t, s = 1, ..., N.$$

Moreover, $\tilde{v}_t$ are to be independent of any $\tilde{\epsilon}_s$ for $s \leqslant t$ and, equally, $\tilde{\epsilon}_t$ are independent of any $\tilde{v}_s$ for $s \leqslant t$. Note that the assumption of linear observations is the characteristically assumption of the estimation theory. This state-observation model finds wide application in the practice. Our goal is to estimate the state vector $\tilde{\beta}_t$ given observations $\tilde{y}_1, ..., \tilde{y}_t$. This estimates are bound to be optimal from the standpoint of robustness. Using the obtained robust estimate we will solve the following optimal control problem

$$\text{minimize } J := \frac{1}{2}\mathrm{E}\Big[\sum_{t=0}^{N-1}(\tilde{\beta}_t', u_t)\Theta_t(\tilde{\beta}_t', u_t)'\Big] \tag{13}$$

$$\text{subject to } (10) - (12), \tag{14}$$

where $\Theta_t := \Theta(\tau_t)$. Given the linear observation equation (12) the discrete optimal control problem (13)–(14) approximates the original optimal control problem (8)–(9).

3. Optimal Robust Estimation

Now we examine the problem of optimal robust estimation of the state vector more generally.

For $t \in \mathbb{N}$ let v_t be independent, identically distributed (i.i.d.) random variables with values in $\mathbb{R}^p$, and, similarly, let $\epsilon_t \in \mathbb{R}^q$ be i.i.d.. These v_t are to be independent of any ϵ_s for $s \leqslant t$ and, equally, ϵ_t independent of any v_s for $s \leqslant t$. Furthermore, let β_0 be an initial random variable in $\mathbb{R}^p$, independent of any v_t, ϵ_t. Given some sequences of matrices $F_t \in \mathbb{R}^{p \times p}$ and $Z_t \in \mathbb{R}^{q \times p}$ for $t \in \mathbb{N}$, we introduce, again for $t \in \mathbb{N}$, a discrete dynamic system as

$$\beta_t = F_t\beta_{t-1} + H_t u_t + v_t, \tag{15}$$

$$y_t = Z_t\beta_t + \epsilon_t, \quad t \in \mathbb{N}, \tag{16}$$

where

$$\mathrm{E}(v_t) = 0, \ \mathrm{E}(\epsilon_t) = 0, \ u_0 = 0, \ \mathrm{E}(\beta_0) = a_0, \ \mathrm{E}(\beta_0\beta_0') = Q_0,$$

$$\mathrm{E}\begin{pmatrix} v_t \\ \epsilon_t \end{pmatrix}(v_k' \ \ \epsilon_k') = \begin{pmatrix} Q_t\delta_{tk} & 0 \\ 0 & V_t\delta_{tk} \end{pmatrix}, \quad k = 1, 2, ...,$$

and δ_{tk} is the Kronecker delta. Let $H_t \in \mathbb{R}^{p \times m}$. By $u_t \in \mathbb{R}^m$, $t \in \mathbb{N}$ we denote a control vector. The matrices F_t, Z_t, H_t, Q_t, V_t are supposed to be known. Let us consider the following optimal control problem

$$\text{minimize } J := \frac{1}{2}\mathrm{E}\Big[\sum_{t=0}^{N-1}(\beta_t', u_t')\begin{pmatrix} A_t & B_t \\ B_t' & C_t \end{pmatrix}(\beta_t', u_t')' + \beta_N'P_N\beta_N\Big] \tag{17}$$

$$\text{subject to } (15) - (16), \tag{18}$$

where $P_N, A_t \in \mathbb{R}^{p \times p}$, $B_t \in \mathbb{R}^{p \times m}$, $C_t \in \mathbb{R}^{m \times m}$. Under the usual assumptions of normality $\beta_0 \sim \mathcal{N}(a_0, Q_0)$, $v_t \sim \mathcal{N}(0, Q_t)$, $\epsilon_t \sim \mathcal{N}(0, V_t)$, the solution of (17)–(18) is given in the form of linear feedback (Bryson and Ho, 1975)

$$u_t^* = -k_t^1 \beta_{t|t}^0, \quad t = 1, ..., N$$

where k_t^1 is an amplification factor and $\beta_{t|t}^0 := \mathrm{E}(\beta_t | y_t^*)$, $y_t^* := (y_1', ..., y_t')'$ is the Kalman estimate of the vector β_t.

It is well known that Kalman filtering is one of the most important developments of the linear estimation theory. It is a useful instrument for recursively treating of controllable dynamic systems (Anderson and Moore, 1979; Bryson and Ho, 1975; Polyak and Tsypkin, 1979). The Kalman recursion is widely used for control of discrete stochastic systems (Bryson and Ho, 1975; Polyak and Tsypkin, 1979; Liptser and Shiryayev, 1977). The standard Kalman filter is given by the following recursion.

- *Initial step:*

$$\beta_{0|0}^0 = a_0, \quad \Sigma_{0|0}^0 = Q_0.$$

- *Prediction:*

$$\beta_{t|t-1}^0 = F_t \beta_{t-1|t-1}^0 + H_t u_{t-1},$$
$$\Sigma_{t|t-1}^0 = F_t \Sigma_{t-1|t-1}^0 F_t' + Q_t, \quad t \in \mathbb{N}$$

  where $\beta_{t|t-1}^0 = \mathrm{E}(\beta_t | y_{t-1}^*)$, $\Sigma_{t|t-1}^0 = \mathrm{Cov}(\beta_t - \beta_{t|t-1}^0)$, $\Sigma_{t|t}^0 = \mathrm{Cov}(\beta_t - \beta_{t|t}^0)$.
- *Correction:*

$$\beta_{t|t}^0 = \beta_{t|t-1}^0 + \hat{M}_t(y_t - Z_t \beta_{t|t-1}^0),$$
$$\Sigma_{t|t}^0 = \Sigma_{t|t-1}^0 - \hat{M}_t Z_t \Sigma_{t|t-1}^0,$$

  where $\hat{M}_t = \Sigma_{t|t-1}^0 Z_t' [Z_t \Sigma_{t|t-1}^0 Z_t' + V_t]^{-1}$.

The Kalman filter assures optimal least-squares estimation of a process only under the assumption that the random perturbations are Gaussian (Anderson and Moore, 1979). If, however, the measurement or state errors contain outliers, a Kalman filter estimator may be quite inaccurate. The corresponding solution of the optimal control problem (17)–(18) can be ill-defined. Therefore, we use a robust version of the Kalman filter for effective solving the optimal control problem (17)–(18) in the non-Gaussian case.

Assume that $\beta_0 \sim G_0(a_0, Q_0)$ and $\mathrm{E}(\beta_0) = a_0$, $\mathrm{E}(\beta_0 \beta_0') = Q_0$, where G_0 is a distribution function of β_0. Let

$$\epsilon_t \sim G \in \mathcal{P}(0, W_t) := \left\{ G \mid \int_{\mathbb{R}^q} \epsilon_t G(d\epsilon_t) = 0, \int_{\mathbb{R}^q} \epsilon_t \epsilon_t' G(d\epsilon_t) \leqslant W_t < \infty \right\}$$

and

$$v_t \sim G \in \mathcal{P}(0, R_t) := \left\{ G \mid \int_{\mathbb{R}^p} v_t G(dv_t) = 0, \int_{\mathbb{R}^p} v_t v_t' G(dv_t) \leqslant R_t < \infty \right\},$$

where W_t, R_t are symmetric, positive semi-definite matrices, and G is a distribution of vector ϵ_t or v_t. For the sequel let us introduce the expressions

$$\Delta\beta_t := \beta_t - \beta_{t|t-1}, \ \beta_{t|t-1} := \mathrm{E}[\beta_t | y_{t-1}^*],$$
$$\Delta y_t := y_t - \mathrm{E}[y_t | y_{t-1}^*], \quad t = 1, 2, ...,$$

and note that $\Delta\beta_1 = F_1(\beta_0 - a_0) + v_1$, $\Delta y_t = Z_t \Delta\beta_t + \epsilon_t$. The random variables $\Delta\beta_t$, $t = 1, 2, ...$ are called innovations. By x_t we denote the vector $(\Delta\beta_t', \Delta y_t')'$ with covariance matrix

$$\mathrm{Cov}(x_t) = \begin{pmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1} Z_t' \\ Z_t \Sigma_{t|t-1} & Z_t \Sigma_{t|t-1} Z_t' + V_t \end{pmatrix}.$$

Suppose that $\mathrm{Cov}(x_t) \leqslant S_t < \infty$, where S_t is a symmetric, positive semi-definite matrix. Here the purpose is to estimate the state β_t given observations y_t^*. The corresponding estimate will be called $\beta_{t|t}$. Using this estimate we solve the optimal control problem (17)–(18). The desired estimate $\beta_{t|t}$ must be defined as a robust estimate.

A formal definition of robustness can be found in Huber (1981). We assume that the observed data is contaminated. One of the most frequent types of contaminated data is ϵ-contaminated normal data with distribution

$$G = (1 - \varsigma)\mathcal{N}(a, \sigma) + \varsigma\mathcal{H}, \quad 0 \leqslant \varsigma \leqslant 1,$$

where $\mathcal{H}$ is a symmetric distribution. Note that the sets $\mathcal{P}(0, W_t)$ and $\mathcal{P}(0, R_t)$ don't specify any topological neighborhood in the space of distributions (Huber, 1971). In practice, however, contaminated data is often described with help of the sets $\mathcal{P}(0, W_t)$, $\mathcal{P}(0, R_t)$. We use the following concept: an estimator is robust if it remains finite as one or more of the data points become arbitrarily large.

Let

$$\Phi := \left\{ g_t : \ \mathbb{Z}_+ \times \mathbb{R}^q \to \mathbb{R}^p \ \middle| \ g_t \text{ is a measurable function} \right\}.$$

Now let us formulate the following auxiliary minimax problem

$$\sup_{\mathcal{P}(0, S_t)} \mathrm{E}||\Delta\beta_t - g_t(\Delta y_t)\Delta y_t||^2 \to \min_{g_t \in \Phi}!, \tag{19}$$

$$\mathcal{P}(0, S_t) := \left\{ G \mid \int_{\mathbb{R}^{p+q}} x_t G(dx_t) = 0, \int_{\mathbb{R}^{p+q}} x_t x_t' G(dx_t) \leqslant S_t < \infty \right\},$$

where G is a distribution of vector x_t. We call the function $g_t(\Delta y_t) \in \Phi$ a weight function. Expression (19) defines a weighted least-squares estimate for $\Delta\beta_t$. If a solution of

the optimization problem (19) exists then we also can get an estimate $\beta_{t|t}$ of the vector β_t as a recursive weighted least-squares estimate. We call this estimate the optimal robust estimate of vector β_t. This estimate is a weighted least squares estimate of $\Delta\beta_t$. Note that the weighted least squares method is one of the most popular algorithms for calculating robust estimators (Huber, 1981; Cipra and Romera, 1991; Bosov and Pankov, 1992; Fahrmeir and Kaufmann, 1991; Pupeikis, 1998; Azhmyakov, 2000).

REMARK 1. The classical Kalman filter is the solution of the problem

$$\mathrm{E}||\Delta\beta_t^0 - M_t\Delta y_t^0||^2 \to \min_{M_t}!,$$

$$M_t \in \mathbb{R}^{p\times q}, \quad t \in \mathbb{N},$$

under the usual assumptions of normality. The weight function is equal to M_t. It is evident that this problem is analogous to (19).

Let

$$S_t := \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

We assume that S_{22} is an invertible matrix. Now we present our main result relative to the optimal robust estimate.

Theorem 2. *Suppose that* $\mathrm{E}(u_t|y_t^*) = u_t,\ t \in \mathbb{N}$. *The problem* (19) *has an unique solution and the optimal robust estimate* $\beta_{t|t}$ *of the vector* β_t *is given by the recursions*

$$\beta_{t|t} = \beta_{t|t-1} + g_t^{\mathrm{opt}}(\Delta y_t)\Delta y_t, \quad \beta_{0|0} = a_0, \tag{20}$$

$$\beta_{t|t-1} = F_t\beta_{t-1|t-1} + H_t u_t, \tag{21}$$

$$g_t^{\mathrm{opt}}(\Delta y_t) = S_{12}S_{22}^{-1}, \quad t \in \mathbb{N}. \tag{22}$$

Proof. First let $x_t^0 \sim \mathcal{N}(0, S_t^0),\ S_t^0 \leqslant S_t$, where

$$S_t^0 = \begin{pmatrix} \Sigma_{t|t-1}^0 & \Sigma_{t|t-1}^0 Z_t' \\ Z_t\Sigma_{t|t-1}^0 & Z_t\Sigma_{t|t-1}^0 Z_t' + V_t \end{pmatrix}.$$

Then the following relation

$$\mathrm{E}||\Delta\beta_t^0 - g_t^0(\Delta y_t^0)\Delta y_t^0||^2 = \min_{g_t \in \Phi} \mathrm{E}||\Delta\beta_t^0 - g_t(\Delta y_t^0)\Delta y_t^0||^2$$

is satisfied, where

$$g_t^0(\Delta y_t^0) := \hat{M}_t = \Sigma_{t|t-1}^0 Z_t'(Z_t\Sigma_{t|t-1}^0 Z_t' + V_t)^{-1},$$

$$g_t^0(\Delta y_t^0)\Delta y_t^0 = \mathrm{E}[\Delta\beta_t^0|y_t^*],$$

and $g_t^0 \in \Phi$. Let

$$J(S_t^0) := \mathrm{E}||\Delta\beta_t^0 - g_t^0(\Delta y_t^0)\Delta y_t^0||^2.$$

Hence $J(S_t^0) = \mathrm{tr}(\Sigma_{t|t}^0)$. Now we consider a vector $x_t \sim G \in \mathcal{P}(0, S_t)$. Then for all $g_t \in \Phi$, such that $\mathrm{E}||g_t(\Delta y_t)\Delta y_t||^2 < \infty$ the inequality

$$\sup_{\mathcal{P}(0,S_t)} \mathrm{E}||\Delta\beta_t - g_t(\Delta y_t)\Delta y_t||^2 \geqslant \mathrm{E}||\Delta\beta_t^0 - g_t(\Delta y_t^0)\Delta y_t^0||^2$$
$$\geqslant \min_{g_t \in \Phi} \mathrm{E}||\Delta\beta_t^0 - g_t(\Delta y_t^0)\Delta y_t^0||^2 = \mathrm{E}||\Delta\beta_t^0 - g_t^0(\Delta y_t^0)\Delta y_t^0||^2 = J(S_t^0)$$

is satisfied. For all $S_t^0 \leqslant S_t$ we obtain the inequality

$$\min_{g_t \in \Phi} \sup_{\mathcal{P}(0,S_t)} \mathrm{E}||\Delta\beta_t - g_t(\Delta y_t)\Delta y_t||^2 \tag{23}$$
$$\geqslant \min_{g_t \in \Phi} \mathrm{E}||\Delta\beta_t^0 - g_t(\Delta y_t^0)\Delta y_t^0||^2 = J(S_t^0).$$

Let us take the matrix $\tilde{S}^0 := \mathrm{argmax}_{S_t^0 \leqslant S_t} J(S_t^0)$ as S_t^0. Denote

$$\tilde{J} := \sup_{S_t^0 \leqslant S_t} J(S_t^0),$$

where

$$\Sigma_{t|t}^0 \leqslant S_{11} - S_{12}S_{22}^{-1}S_{21}.$$

Evidently the trace $\mathrm{tr}(\Sigma_{t|t}^0)$ of matrix $\Sigma_{t|t}^0$ is a non-decreasing function (the matrix $\Sigma_{t|t}^0$ is a positive semi-definite matrix). Therefore $\tilde{S}^0 = S_t$ and

$$\tilde{J} = J(S_t).$$

Now inequality (23) can be written as

$$\min_{g_t \in \Phi} \sup_{\mathcal{P}(0,S_t)} \mathrm{E}||\Delta\beta_t - g_t(\Delta y_t)\Delta y_t||^2 \geqslant \tilde{J}, \tag{24}$$

where

$$\tilde{J} = \mathrm{tr}(\mathrm{Cov}(\Delta\beta_t^0 - \tilde{g}_t(\Delta y_t^0)\Delta y_t^0)) = \mathrm{tr}(S_{11} - S_{12}S_{22}^{-1}S_{21}),$$
$$\tilde{g}_t(\Delta y_t^0) := S_{12}S_{22}^{-1}.$$

It is obvious that $\tilde{g}_t \in \Phi$.

For the vector $x_t \sim G \in \mathcal{P}(0, S_t)$ with a covariance matrix $\mathrm{Cov}(x_t) \leqslant S_t$ we get

$$\mathrm{E}||\Delta\beta_t - g_t^0(\Delta y_t)\Delta y_t||^2 = \mathrm{tr}(\Sigma_{t|t-1} - \Sigma_{t|t-1}Z_t'(Z_t\Sigma_{t|t-1}Z_t' + V_t)Z_t\Sigma_{t|t-1})$$
$$= J(S_t^0).$$

Hence

$$\sup_{\mathcal{P}(0,S_t)} \mathrm{E}||\Delta\beta_t - g_t^0(\Delta y_t)\Delta y_t||^2 = \sup_{\mathcal{P}(0,S_t)} \mathrm{tr}(\Sigma_{t|t})$$

$$= \sup_{\mathrm{Cov}(x_t) \leqslant S_t} J(\mathrm{Cov}(x_t)) = J(S_t) = \tilde{J}, \tag{25}$$

and

$$\sup_{\mathcal{P}(0,S_t)} \mathrm{tr}(\Sigma_{t|t}) = \mathrm{E}||\Delta\beta_t - \tilde{g}_t(\Delta y_t)\Delta y_t||^2. \tag{26}$$

From (25) and (26) it follows that the inequality

$$\min_{g_t \in \Phi} \sup_{\mathcal{P}(0,S_t)} \mathrm{E}||\Delta\beta_t - g_t(\Delta y_t)\Delta y_t||^2$$

$$\leqslant \sup_{\mathcal{P}(0,S_t)} \mathrm{E}||\Delta\beta_t - \tilde{g}_t(\Delta y_t)\Delta y_t||^2 = \sup_{\mathcal{P}(0,S_t)} \mathrm{tr}(\Sigma_{t|t}) = \tilde{J} \tag{27}$$

is satisfied.

From relations (24) and (27) we get the relation (22)

$$g_t^{\mathrm{opt}} = S_{12}S_{22}^{-1}.$$

The normal equation for $(\Delta\beta_t - g_t(\Delta y_t)\Delta y_t)$ implies the formulae (20) and (21) for $\beta_{t|t}$ and $\beta_{t|t-1}$. Let $\beta_{0|0} = \beta_0$. The theorem is proved.

COROLLARY 1. The relations for covariance matrices $\Sigma_{t|t-1} := \mathrm{Cov}(\beta_- \beta_{t|t-1})$ and $\Sigma_{t|t} := \mathrm{Cov}(\beta_t - \beta_{t|t})$ are given by the following recursions

$$\Sigma_{t|t-1} = F_t\Sigma_{t-1|t-1}F_t' + Q_t, \tag{28}$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}L_t' - L_t\Sigma_{t|t-1} + L_t\Sigma_{t|t-1}L_t' + S_{12}S_{22}^{-1}V_tS_{22}^{-1}S_{12}', \tag{29}$$

where $\Sigma_{0|0} = Q_0$ and $L_t := S_{12}S_{22}^{-1}Z_t, \ t \in \mathbb{N}$.

This corollary can be proved by direct calculations.

The formulae (20)–(22) and (28)–(29) define the optimal (in the sense of the problem (19)) robust estimate of the vector β_t. As we can see the estimate $\beta_{t|t}$ has the structure of the classical Kalman filter.

4. Some Properties of the Optimal Robust Estimate

In this section we show that the conditional density of the optimal robust estimate (20)–(22) is a Gaussian density. The obtained estimation has a form of "weighted" recursion

$$\beta_{t|t} = \beta_{t|t-1} + w_t(\Delta y_t)\Delta y_t, \quad \beta_{0|0} = a_0, \tag{30}$$

$$\Delta y_t := y_t - \mathrm{E}(y_t | y_{t-1}^*) = Z_t \Delta\beta_t + \epsilon_t, \tag{31}$$

$$\beta_{t|t-1} = F_t \beta_{t-1|t-1} + H_t u_t, \tag{32}$$

$$t \in \mathbb{N},$$

where $w_t(\Delta y_t) = g_t^{opt}(\Delta y_t)$. Many famous variants of robust Kalman filter have the same structure of "weighted" recursion (Fahrmeir and Kaufmann, 1991; Bosov and Pankov, 1992; Cipra and Romera, 1991; Azhmyakov, 2000).

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space of the processes y_t and Δy_t. By $\mathcal{F}_t^{y_t}$ and $\mathcal{F}_t^{\Delta y_t}$ we denote σ-algebras generated by y_t and Δy_t respectively.

Lemma 1. *For the processes y_t and Δy_t defined by formulae* (16) *and* (31) *the equality*

$$\mathcal{F}_t^{y_t} = \mathcal{F}_t^{\Delta y_t}, \quad \forall t > 1$$

is satisfied.

Proof. By definition we have $\Delta y_t = y_t - Z_t \beta_{t|t-1}$. The functions $\beta_{t|t-1}$ and Δy_t, $t > 1$ are measurable functions of the vector $(y_1', ... y_{t-1}')'$. Hence

$$\mathcal{F}_t^{\Delta y_t} \subseteq \mathcal{F}_t^{y_t}, \quad \forall t > 1.$$

From (30)–(32) it follows that $\forall t > 1$ the stochastic function

$$y_t = \Delta y_t + Z_t F_t \beta_{t|t-1}$$

is a $\mathcal{F}_t^{y_t}$-measurable function. Finally we obtain $\mathcal{F}_t^{y_t} \subseteq \mathcal{F}_t^{\Delta y_t}$.

Now we consider the following recursions for innovations

$$\Delta\beta_{t+1} = F_{t+1}(I - w_t(\Delta y_t)Z_t)\Delta\beta_t - F_{t+1}w_t(\Delta y_t)\epsilon_t + v_{t+1}, \quad t \geqslant 0.$$

Let $v_1 \sim \mathcal{N}(0, Q_1)$, $\epsilon_1 \sim \mathcal{N}(0, V_1)$. We assume that there exists a conditional density function $p(\Delta\beta_t | \mathcal{F}_t^{\Delta y_t})$ of the vector $\Delta\beta_t$.

Theorem 3. *The conditional density function $p(\Delta\beta_t | \mathcal{F}_t^{\Delta y_t})$, $t \in \mathbb{N}$ is a Gaussian density function*

Proof. The proof is by induction on t.

For $t = 1$ we obtain

$$\Delta\beta_1 = F_1(\beta_0 - a_0) + v_1,$$

$$\Delta y_1 = Z_1 F_1(\beta_0 - a_0) + Z_1 v_1 + \epsilon_1,$$

and

$$(\Delta\beta_1, \Delta y_1)' \sim N_{p+q}\left(0, \begin{pmatrix} F_1 Q_0 F_1' + Q_1 & (F_1 Q_0 F_1' + Q_1)Z_1' \\ Z_1(F_1 Q_0 F_1' + Q_1) & Z_1(F_1 Q_0 F_1' + Q_1)Z_1' + V_1 \end{pmatrix}\right).$$

It follows that

$$
\begin{aligned}
& P(\Delta\beta_1 \leqslant \chi \mid \mathcal{F}_1^{\Delta y_1}) \\
& = N_p\big\{0 + (F_1 Q_0 F_1' + Q_1)Z_1'(Z_1(F_1 Q_0 F_1' + Q_1)Z_1' + V_1)^- \times (\Delta y_1 - 0), \\
& \quad (F_1 Q_0 F_1' + Q_1) - (F_1 Q_0 F_1' + Q_1)Z_1'(Z_1(F_1 Q_0 F_1' + Q_1)Z_1' + V_1)^- \\
& \quad \times Z_1(F_1 Q_0 F_1' + Q_1)\big\},
\end{aligned}
$$

where $\chi \in \mathbb{R}^p$.

Let $P(\Delta\beta_s \leqslant \chi \mid \mathcal{F}_s^{\Delta y_s}) = \mathcal{N}_p(m_s, \gamma_s)$. We use the Lemma 1 and consider the following conditional distribution function $P(\Delta\beta_s \leqslant \chi \mid \mathcal{F}_s^{y_s}) = \mathcal{N}_p(m_s, \gamma_s)$. Then we have

$$P(\Delta\beta_{s+1} \leqslant \chi, \ \Delta y_{s+1} \leqslant \varphi \mid \Delta\beta_s, \ \mathcal{F}_s^{y_s}) = \mathcal{N}(A, B), \quad \varphi \in \mathbb{R}^q,$$

where

$$
\begin{aligned}
A = (c_1, c_2)' & = \begin{pmatrix} F_{s+1} \\ Z_{s+1}F_{s+1} \end{pmatrix} \Delta\beta_s - \begin{pmatrix} F_{s+1}w_s(\Delta y_s) \\ Z_{s+1}F_{s+1}w_s(\Delta y_s) \end{pmatrix} \Delta y_s \\
& = A_1 \Delta\beta_s - A_2(\Delta y_s), \\
c_1 \in \mathbb{R}^p, & \quad c_2 \in \mathbb{R}^q,
\end{aligned}
$$

and the covariance matrix B is equal to

$$B = \begin{pmatrix} Q_{s+1} & 0 \\ 0 & V_{s+1} \end{pmatrix}.$$

Note that $(\Delta\beta_{s+1} - c_1) = v_{s+1}, \ (\Delta y_{s+1} - c_2) = \epsilon_{s+1}$.

Now we consider the characteristic function of the vector $(\Delta'\beta_{s+1}, \ \Delta'y_{s+1})'$

$$
\begin{aligned}
& \mathrm{E}(\exp(i\lambda'(\Delta'\beta_{s+1}, \ \Delta'y_{s+1})')|\Delta\beta_s, \mathcal{F}_s^{y_s}) \\
& = \exp[i\lambda'(A_1\Delta\beta_s - A_2(\Delta y_s)) - 0.5\lambda'B\lambda], \quad \lambda \in \mathbb{R}^{p+q}.
\end{aligned}
$$

By induction step we obtain

$$\mathrm{E}(\exp(i\lambda'A_1\Delta\beta_s)|\mathcal{F}_s^{y_s}) = \exp[i\lambda'A_1 m_s - 0.5\lambda'A_1\gamma_s A_1'\lambda].$$

Therefore the relation

$$
\begin{aligned}
& \mathrm{E}(\exp(i\lambda'(\Delta'\beta_{s+1}, \ \Delta'y_{s+1})')|\mathcal{F}_s^{y_s}) \\
& = \exp\big[i\lambda'(A_1 m_s - A_2(\Delta y_s)) - 0.5\lambda'(B + A_1\gamma_s A_1')\lambda\big],
\end{aligned}
$$

and

$$P(\Delta\beta_{s+1} \leqslant \chi, \ \Delta y_{s+1} \leqslant \varphi \mid \mathcal{F}_s^{y_s}) = N(A_1 m_s - A_2(\Delta y_s), \ B + A_1\gamma_s A_1'),$$
$$\varphi \in \mathbb{R}^q$$

are satisfied. It follows that the conditional distribution function

$$P\big(\Delta\beta_{s+1} \leqslant \chi \mid \mathcal{F}_{s+1}^{y_{s+1}}\big)$$

is also Gaussian distribution function (Liptser and Shiryayev, 1977).

Thus the innovations $\Delta\beta_t, \ t = 1, 2, \ldots$ are conditionally normal random variables. By m_t and $\gamma_t, \ t \in \mathbb{N}$ we denote the conditional expectation and the conditional covariance matrix of the process $\Delta\beta_t$. It is evident that

$$
\begin{aligned}
m_1 &= (F_1 Q_0 F_1' + Q_1) Z_1'(Z_1(F_1 Q_0 F_1' + Q_1)Z_1' + V_1)^- \Delta y_1, \\
\gamma_1 &= (F_1 Q_0 F_1' + Q_1) \\
&\quad - (F_1 Q_0 F_1' + Q_1) Z_1'(Z_1(F_1 Q_0 F_1' + Q_1)Z_1' + V_1)^- Z_1(F_1 Q_0 F_1' + Q_1).
\end{aligned}
$$

COROLLARY 2. For the parameters m_t and γ_t of the density function $p(\Delta\beta_t|\mathcal{F}_t^{y_t})$ the following relations

$$
\begin{aligned}
m_{t+1} &= (F_{t+1} - \Xi_{t+1}Z_t)m_t - \Xi_{t+1}\Delta y_t^0 \\
&\quad + \Upsilon_{t+1}(\Delta y_{t+1} - Z_{t+1}((F_{t+1} - \Xi_{t+1}Z_t)m_t - \Xi_{t+1}\Delta y_t^0)), \qquad (33) \\
\gamma_{t+1} &= (F_{t+1}\gamma_t F_{t+1}' + Q_{t+1}) - \Upsilon_{t+1}Z_{t+1}(F_{t+1}\gamma_t F_{t+1}' + Q_{t+1}), \qquad (34) \\
\Upsilon_{t+1} &= (F_{t+1}\gamma_t F_{t+1}' + Q_{t+1})Z_{t+1}' \times [Z_{t+1}(F_{t+1}\gamma_t F_{t+1} + Q_{t+1})Z_{t+1}' + V_{t+1}]^-, \\
\Xi_{t+1} &:= F_{t+1}w_t(\Delta y_t), \quad t \in \mathbb{N}
\end{aligned}
$$

are satisfied.

Proof. We have

$$
\begin{aligned}
m_{t+1} &= \mathrm{E}(\Delta\beta_{t+1}|\mathcal{F}_t^{y_t}) + \Upsilon_{t+1}(\Delta y_{t+1} - \mathrm{E}(\Delta y_{t+1}|\mathcal{F}_t^{y_t})), \\
\gamma_{t+1} &= d_1 - \Upsilon_{t+1}d_2,
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &:= \mathrm{Cov}(\Delta\beta_{t+1}|\mathcal{F}_t^{y_t}), \\
d_2 &:= \mathrm{Cov}(\Delta y_{t+1}, \Delta\beta_{t+1}|\mathcal{F}_t^{y_t}),
\end{aligned}
$$

and

$$\Upsilon_{t+1} = \mathrm{Cov}(\Delta\beta_{t+1}, \Delta y_{t+1}|\mathcal{F}_t^{y_t})\mathrm{Cov}(\Delta y_{t+1}|\mathcal{F}_t^{y_t})^-.$$

Thus we obtain the formula

$$m_{t+1} = (F_{t+1} - \Xi_{t+1}Z_t)m_t - \Xi_{t+1}\Delta y_t^0$$
$$+ \Upsilon_{t+1}(\Delta y_{t+1} - Z_{t+1}((F_{t+1} - \Xi_{t+1}Z_t)m_t - \Xi_{t+1}\Delta y_t^0)),$$
$$t \in \mathbb{N}.$$

For the vectors $\Delta\beta_t$ and Δy_t the following relations

$$\Delta\beta_{t+1} - \mathrm{E}(\Delta\beta_{t+1}|\mathcal{F}_t^{y_t}) = F_{t+1}(\Delta\beta_t - m_t) + v_{t+1},$$
$$\Delta y_{t+1} - \mathrm{E}(\Delta y_{t+1}|\mathcal{F}_t^{y_t}) = Z_{t+1}F_{t+1}(\Delta\beta_t - m_t) + Z_{t+1}v_{t+1} + \epsilon_{t+1},$$

are satisfied.

Now we calculate the covariance matrices d_1 and d_2

$$d_1 = F_{t+1}\gamma_t F_{t+1}' + Q_{t+1},$$
$$d_2 = Z_{t+1}(F_{t+1}\gamma_t F_{t+1}' + Q_{t+1}),$$
$$\Upsilon_{t+1} = d_2'(Z_{t+1}(F_{t+1}\gamma_t F_{t+1}' + Q_{t+1})Z_{t+1}' + V_{t+1})^-,$$

and at last we obtain the formulae (33)–(34).

The conditionally normality of the vectors $\Delta\beta_t$ makes possible more detailed investigation of the robust estimate (20)–(22).

5. Optimal Linear Feedback

For solving the optimal control problems (17)–(18) and (13)–(14) for $t = 1, ..., N$ we use the optimal robust estimate given by formulae (20)–(22). First let us consider the problem (17)–(18). We construct the optimal control u_t as a linear function of the vector $\beta_{t|t}$ (linear feedback) (Bryson and Ho, 1975) namely

$$u_t^* = -k_t^2\beta_{t|t}, \ t = 1, ..., N, \tag{35}$$

where the matrix k_t^2 is equal to

$$k_t^2 = (H_t'P_{t+1}H_t + C_t)^{-1}(H_t'P_{t+1}F_t + B_t'), \tag{36}$$
$$P_t = F_t'P_{t+1}F_t - (k_t^2)'(C_t + H_t'P_{t+1}H_t)k_t^2 + A_t. \tag{37}$$

The matrix P_N is given. The optimal robust estimate (20)–(22) has the structure of the classical Kalman filter. Therefore using the optimality principle of Bellman one can prove (Bryson and Ho, 1975) that the optimal control function has the form of the linear feedback (35).

The formulae for the optimal robust estimate (20)–(22), the recursions for covariance matrices (28)–(29) and relations (36)–(37) completely determine the solution (35) of the

problem (17)–(18). This solution is robust with respect to stochastic perturbations (outliers). It is evident that the condition $E(u_t|y_t^*) = u_t, \ t = 1, 2, ..., N$ of the Theorem 2 is also satisfied.

The covariance matrix $\Sigma_{t|t-1}$ satisfies the algebraic Riccati equation (Anderson and Moore, 1979)

$$\Sigma_{t+1|t} = F_t(\Sigma_{t|t-1} - \Sigma_{t|t-1}Z_t'[Z_t\Sigma_{t|t-1}Z_t' + V_t]^{-1}Z_t\Sigma_{t|t-1})F_t' + Q_t.$$

Now we use the presented theory of the optimal robust estimation in the discrete optimal control problem of a bioreactor (13)–(14). Consider the model (10)–(12). It was assumed that

$$\tilde{\epsilon}_t \sim G \in \mathcal{P}(0, W_t) := \left\{ G \mid \int_{\mathbb{R}^2} \tilde{\epsilon}_t G(d\tilde{\epsilon}_t) = 0, \ \int_{\mathbb{R}^2} \tilde{\epsilon}_t\tilde{\epsilon}_t'G(d\tilde{\epsilon}_t) \leqslant W_t < \infty \right\},$$

and

$$\tilde{v}_t \sim G \in \mathcal{P}(0, R_t) := \left\{ G \mid \int_{\mathbb{R}^2} \tilde{v}_t G(d\tilde{v}_t) = 0, \ \int_{\mathbb{R}^2} \tilde{v}_t\tilde{v}_t'G(d\tilde{v}_t) \leqslant R_t < \infty \right\}.$$

For the problem (13)–(14) we have $P_N \equiv 0$ and

$$\tilde{F}_t := (I - \Delta\tau a_1(\tau_t))^{-1}, \ \tilde{H}_t := (I - \Delta\tau a_1(\tau_t))^{-1}a_2(\tau_t).$$

Let

$$\Theta_t = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{pmatrix}.$$

The optimal solution u_t^{opt} of the approximating problem (13)–(14) is given by following relations

$$u_t^{opt} = -k_t^3\tilde{\beta}_{t|t}, \quad t = 1, ..., N , \tag{38}$$

$$k_t^3 = \frac{\tilde{H}_t'P_{t+1}\tilde{F}_t + (\theta_{31}, \theta_{32})}{\tilde{H}_t'P_{t+1}\tilde{H}_t + \theta_{33}}, \tag{39}$$

$$P_t = \tilde{F}_t'P_{t+1}F_t - (k_t^3)'(\theta_{33} + \tilde{H}_t'P_{t+1}\tilde{H})k_t^3 + \tilde{A}_t, \tag{40}$$

$$P_N = 0, \tag{41}$$

where

$$\tilde{A}_t := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}.$$

The vector $\tilde{\beta}_{t|t}$ is defined by the recursions (20)–(22) and (28)–(29). These recursions and the formulae (38)–(41) completely define the optimal solution of the discrete optimal

control problem (13)–(14). The solution of the optimal control problem problem (13)–(14) is given by linear feedback (38).

References

Anderson, B.O, J.B. Moore (1979). *Optimal Filtering*. Prentice–Hall, Inc., Englewood Clifs, New Jersey.

Azhmyakov, V. (2000). Minimax robust estimation and control in stochastic systems. *Stability and Control: Theory and Appl.*, **3**, 253–262.

Bailey, J., D. Ollis (1986). *Biochemical Engineering Fundamentals*. McGraw–Hill, New York.

Bernard, M., P. Guy (1988). Parameters estimation of an aquatic biological system by the adjoint method. *Math. and Comput. Simul.*, **30**, 405–418.

Bosov, A.V., A.R. Pankov (1992). Robust recursive estimation of processes in stochastic systems. *Avtomat. i Telemekh.*, **9**, 102–110 (in Russian).

Bryson, A.E., J.C. Ho (1975). *Applied Optimal Control*. J. Wiley, New York.

Cipra, T., R. Romera (1991). Robust Kalman filter and its application in time series analysis. *Kybernetika*, **27**, 481–494.

Clark, C.W. (1990). *Mathematical Bioeconomics: The Optimal Management of Renewable Resourses*. Wiley–Interscience, New York.

Dochain, D., G. Bastin (1990). Adaptive control of fedbatch bioreactors. *Chem. Eng. Comm.*, **87**, 67–85.

Fahrmeir, L., H. Kaufmann (1991). On Kalman filtering, posterior mode estimation and Fisher scoring in dynamic exponential family regression. *Metrika*, **38**, 37–60.

Heinricher, A., S. Lenhart, A. Solomon (1995). The application of optimal control methodology to a well-stirred bioreactor. *Natural Resource Modeling*, **9**, 61–80.

Huber P.J. (1981). *Robust Statistics*. Weley, New York.

Lenhart, S.M., T.I. Seidman, J. Yong (2001). Optimal control of a bioreactor with modal switching. *Math. Models and Methods in Appl. Sciences*, **11**, 933–949.

Liptser, R.S., A.N. Shiryayev (1977). *Statistics of Random Processes*. Springer, New York.

Polyak, B.T., Ya.Z. Tsypkin (1979). Adaptive estimation algorithms: convergence, optimality, stability. *Avtomath. i Telemekh.*, **3**, 71–84 (in Russian).

Pupeikis, R. (1998). State estimation of dynamic systems in the presence of time-varying outliers in observations. *Informatica*, **9**, 325–342.

V. Azhmyakov has graduated from Moscow Technical University in 1989. He received Ph.D. degree (Applied Mathematics) at the Institute for Control Problems of Russian Academy of Sciences (Moscow) in 1994. In 1997 he was invited as a research assistent to the University of Bayreuth (Germany). Recently he is an Associate Professor at the EMA University of Greifswald (Germany), Department of Numerical Analysis. His research interests include optimization theory, nonlinear analysis and applications, numerical methods in optimal control, signal processing.

Optimalus gerai maišomo bioreaktoriaus valdymas esant stochastiniams trikdžiams

Vadim AZHMYAKOV

Straipsnyje nagrinėjamas biomediacijos bioreaktoriuje matematinis modelis esant idealiam medžiagų maišymuisi. Tiriamos sistemos dinamika yra aprašoma stochastinėmis diferencialinėmis lygtimis. Sprendžiamas optimalaus valdymo uždavinys su kvadratine tikslo funkcija naudojant gerai maišomo bioreaktoriaus linearizuotą modelį. Optimalus valdymas grindžiamas optimaliais robastiniais būklės įverčiais. Apytikris sprendinys gaunamas kaip tiesinis grįžtamasis ryšys.